A Vizing-like theorem for union vertex-distinguishing edge coloring

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Extended Abstract

A vertex-distinguishing coloring of a graph G consists in an edge or a vertex coloring (not necessarily proper) of G leading to a labeling of the vertices of G, where all the vertices are distinguished by their labels.

There are several possible rules for both the coloring and the labeling. For instance, in a *set irregular edge coloring* [5], the label of a vertex is the union of the colors of its adjacent edges. Other rules for the labeling of a vertex from an edge coloring have been studied: the multiset of its adjacent colors [1], their sum [3], product or difference [6] (for those three rules, the colors must be integers)... The variant where the edge coloring is proper has also been studied [2]. If the vertices are colored, we can define the *identifying coloring* [4], in which each vertex is assigned a label corresponding to the union of its closed neighbourhood colors.

Motivated by a generalization of the set irregular edge coloring problem, we introduce a variant of the problem: to each edge is associated a nonempty set of colors. Given a simple graph G, a *k-coloring* of G is a function $f : E(G) \to 2^{\{1\dots,k\}}$ where every edge is labeled using a non-empty subset of $\{1, \dots, k\}$. For any k-coloring f of G, we define, for every vertex u, the set $id_f(u)$ as follows:

$$id_f(u) = \bigcup_{v \text{ s.t. } uv \in E} f(uv).$$

If the context is clear, we will simply write $id(u)$ for $id_f(u)$. A k-coloring f is *union vertex-distinguishing* if, for all distinct u, v in $V(G)$, $id(u) \neq id(v)$. Figure 1 shows an example of such a 4-coloring. For a given graph G, we denote by $\chi_\cup(G)$ the smallest integer such that there exists a union vertex-distinguishing coloring of G. The union vertex-distinguishing edge coloring being defined only on graphs without any connected component of size 1 or 2, we will only consider such graphs. It is easy to notice that such graphs admit a valid union vertex-distinguishing edge coloring.

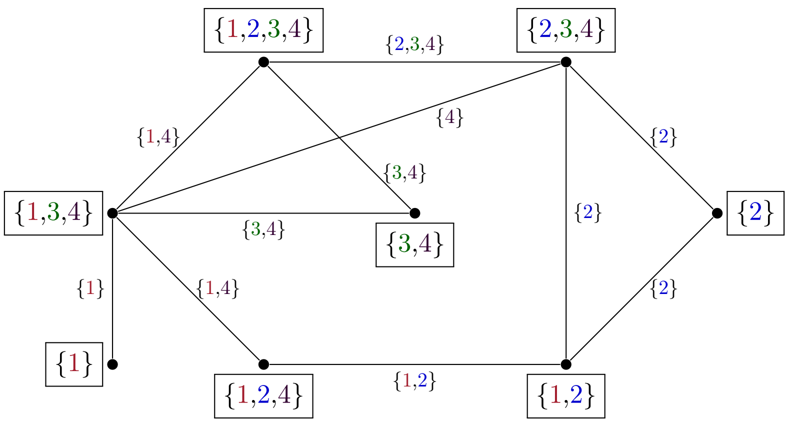

Figure 1: An example of a vertex-distinguishing edge coloring.

We have both lower and upper bounds for $\chi_\cup$:

Proposition 1. *For any graph G with no connected component of size 1 or 2, we have the following inequalities:*

$$\lceil \log_2(|V(G)| + 1) \rceil \leq \chi_\cup(G) \leq \min(\chi_S(G), \chi_{id}(G))$$

where $\chi_S(G)$ is the set irregular edge coloring number of G, and $\chi_{id}(G)$ is the identifying number of G.

The lower bound comes from the fact that, from k colors, one can have at most $2^k - 1$ labels, since labels are nonempty subsets of $\{1, \ldots, k\}$. The upper bound comes from the relationship between our parameter $\chi_\cup$ and other vertex-distinguishing parameters: a set irregular edge coloring is a union vertex-distinguishing edge coloring where only singletons are allowed, and any identifying coloring induces a valid union vertex-distinguishing edge coloring.

We say that a graph G is *optimally colored* if $\chi_\cup(G) = \lceil \log_2(|V| + 1) \rceil$. For example, the graph shown on Figure 1 is optimally colored, since it has 8 vertices and the coloring uses 4 colors. We will see that some classes of graphs can be optimally colored, and that any graph with no connected component of size 1 or 2 admits a union vertex-distinguighing edge coloring with at most 2 more colors than the optimal bound. This is the main result of our paper.

Theorem 2. *For any graph G, we have the following property:*

$$\lceil \log_2(|V(G)| + 1) \rceil \leq \chi_\cup(G) \leq \lceil \log_2(|V(G)| + 1) \rceil + 2$$

Sketch of proof. For any graph G, if H is a graph such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$, then we call H an $edge-subgraph$ of G.

Our proof follows the following schema:

1. We prove that for any edge-subgraph H of a graph G, we have $\chi_\cup(G) \leq \chi_\cup(H) + 1$.

2. We then prove that any graph G admits an edge-subgraph isomorphic to a disjoint union of stars subdivided at most once.

3. We now prove that stars subdivided at most once can be optimally colored.

4. Finally, we prove that a disjoint union of graphs that can be separately optimally colored can be colored together using at most the optimal number of colors plus one.

Thus, Theorem 2 is proved.

We will present sketches of proofs for each item:

The first point is easily proved: if we assign to each edge of $E(G) \backslash E(H)$ a color that has not been used in the coloring of H, then the resulting coloring will be union vertex-distinguishing.

The second point is proved by contradiction. We study the smallest graph G which does not admit a disjoint union of stars subdivided at most once as an edge-subgraph. By minimality, if we take u a vertex of maximum degree, then for every neighbour v of u, the component of v in $G \setminus (u, v)$ is reduced to a single vertex or an edge. This implies that the component of v in $G \setminus (u, v)$, as well as in G, is a star subdivided at most once, a contradiction.

The third point is proved by finding a valid union vertex-distinguishing edge coloring of any star subdivided at most once. There are two cases according to there are $2^k - 1$ vertices of degree 2 in the neighbourhood of the central vertex or not. In both cases, such a coloring can be constructed.

The fourth point is a generalization of a smaller result: let H_1 and H_2 be two graphs such that $2^k \leq |V(H_1)|, |V(H_2)| \leq 2^{k+1} - 1$. If both H_1 and H_2 can be optimally colored, then their disjoint union $H_1 \cup H_2$ can be optimally colored. Indeed, we only need to add the color $k+1$ to each edge of H_2. The resulting coloring will be union vertex-distinguishing and optimal for $H_1 \cup H_2$.

Thus, if we have a family of graphs that can be separately optimally colored, we begin by grouping the graphs H_i, H_j verifying $2^k \leq |V(H_i)|, |V(H_j)| \leq 2^{k+1} - 1$ for a certain k until no graph satisfies this anymore. We then use induction to join the remaining graphs, again by using a color that was not used in any of the graphs that we join. The plus one comes from the fact that if we have two graphs H_i and H_j such that $2^{k_i} \leq |V(H_i)| \leq 2^{k_i+1} - 1 <$

$2^{k_j} \leq |V(H_j)| \leq 2^{k_j+1} - 1$ and verify $|V(H_i)| + |V(H_j)| < 2^{k_j+1}$, then we use $k_j + 1$ colors and thus the coloring is not optimal. □

As seen in the proof of Theorem 2, we actually "lose" a value on the bound by proving the second point. We thus conjecture that the upper bound can be improved:

Conjecture 3. *For any graph G with no connected component of size 1 or 2, we have:*

$$\lceil \log_2(|V(G)| + 1) \rceil \leq \chi_\cup(G) \leq \lceil \log_2(|V(G)| + 1) \rceil + 1$$

There are several possibilities to try and prove or disprove this conjecture: a first idea would be to study the exact value of $\chi_\cup$ for trees or stars subdivided at most once. If graphs from one of these two classes are optimally colorable, then the conjecture is true.

In addition, we proved that paths, cycles and complete binary trees can be optimally colored, which validates the above conjecture for these classes and all the graphs that contain a graph of one of these classes as an edge subgraph (e.g. Hamiltonian graphs).

Theorem 4. *We have $\chi_\cup(G) = \lceil \log_2(|V(G)|+1) \rceil$ if G belongs to one of the following classes of graphs:*

1. *Paths of length greater than 2;*
2. *Cycles of length greater than 3 and different from 7;*
3. *Complete binary trees.*

We have $\chi_\cup(G) = \lceil \log_2(|V(G)| + 1) \rceil + 1$ for the following classes of graphs:

4. *Cycles of length 3 and 7;*
5. *The complete graphs of order $2^k - 1$.*

Sketch of proof. For the first point, we actually prove a slightly larger statement: for $n \geq 3$, there exists an optimal union vertex-distinguishing m-coloring for a path $P_n = (u_1, \ldots, u_n)$ such that $id(u_1) = \{1\}$, $id(u_n) = \{m\}$ and the only vertex that satisfies $id(u_j) = \{1, m\}$ is u_{n-1}. This is proved by induction on n. If $n = 2^k + \ell$ where $0 \leq \ell \leq 2^k - 1$, we use the optimal colorings of P_{2^k-1} and P_ℓ to create an optimal coloring of P_n.

For the second point, we have two cases: either $n = 2^k - 1$ or not. In the latter case, we connect the first and last vertices of the path P_n to create an optimal coloring of C_n. The former case is proved by induction on k.

For the third point, we use induction on the height of the complete binary tree.

For the fourth point, it is easily seen that both C_3 and C_7 can not be optimally colored, but using respectively 3 and 4 colors a valid vertex-distinguishing edge coloring can be obtained.

The fifth point is proven by contradiction: if complete graphs of order $2^k - 1$ could be optimally colored, then two vertices u and v would be identified each by a different singleton $\{i\}$ and $\{j\}$. This is contradictory, since the edge uv would have to be colored with both the singleton $\{i\}$ and the singleton $\{j\}$. □

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