Computability and Complexity

# Exam

## Turing machines and decidability

## Guidelines

This is an oral exam. You have 2 hours to prepare exercises, and then 45 minutes to explain your answers. **You have to solve 1 exercise per part**, so in total you will have to prepare 6 exercises.

## 1 Turing machines

**Exercise 1 : *Logic 101*.**
Give an implementation-level description of a Turing machine that, from an input $\#w_1\#w_2\#$ (where $w_1$ and $w_2$ are strings over $\{0,1\}$, with $|w_1| = |w_2| > 0$), compute **one** of the following functions:

1. The logical AND ($x$ AND $y = 1$ if and only if $x = y = 1$);
2. The logical OR ($x$ OR $y = 0$ if and only if $x = y = 0$);
3. The logical XOR ($x$ XOR $y = 1$ if and only if $x \neq y$).

Those functions are **bit-by-bit** operations. So, for instance, 111 AND 101 = 101; 000 OR 101 = 101; 111 XOR 101 = 010.

□

**Exercise 2 : *ABC is easy as 123*.**
For a word $w$ and a character $\ell$, denote by $|w|_\ell$ the number of $\ell$'s in $w$.
Decide $L = \{w \in \{a,b,c\}^* \mid |w|_a > |w|_b > |w|_c\}$.

□

## 2 Computation models

**Exercise 3 : *Without Left*.**
What class of languages is recognized by a Turing machine where the head can write and then move Right or stay Still, but not move Left? Explain your answer.

□

**Exercise 4 : *Challenge: Resetting*.**
What class of languages is recognized by a Turing machine where the head can write and then move Right or reset to the leftmost cell of the tape, but not move Left? Explain your answer.

□

## 3 Closure under complement

**Exercise 5 :** ***Complementation.***
Recall that the complement of a language $L$ over $\Sigma$ is $\overline{L} = \Sigma^* \setminus L$.

1. Prove that decidable languages are closed under complement.
2. Prove that Turing-recognizable languages are **not** closed under complement.

□

## 4 Other closures

**Exercise 6 :** ***Union.***
Prove that decidable languages **and** Turing-recognizable languages are closed under union.

□

**Exercise 7 :** ***Intersection.***
Prove that decidable languages **and** Turing-recognizable languages are closed under intersection.

□

**Exercise 8 :** ***Concatenation.***
Prove that decidable languages **and** Turing-recognizable languages are closed under concatenation.

□

## 5 Reducibility

**Exercise 9 :** ***United states.***
Prove that the following language is undecidable:

$L = \{< M, q, w > \mid M \text{ is a Turing machine, } q \text{ is a state of } M\text{, and } M \text{ reaches } q \text{ in its computation on } w\}.$

.

□

**Exercise 10 :** ***Inclusivity.***
Prove that the following language is undecidable:

$$L = \{< M_1, M_2 > \mid M_1 \text{ and } M_2 \text{ are Turing machines, and } L(M_1) \subseteq L(M_2)\}.$$

□

## 6 Rice's Theorem

**Exercise 11 :** ***Context-freedom.***
Prove that the following language is undecidable:

$$L = \{< M > \mid M \text{ is a Turing machine, and } L(M) \text{ is context-free}\}.$$

□

**Exercise 12 :** ***Reverting.***
Prove that the following language is undecidable:

$$\{< M > \mid M \text{ is a Turing machine, and } L(M) = L(M)^R\}.$$

□