Computability and Complexity

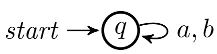
Exercises with corrections

Finite automatas and regular languages

Exercise 1 : ***First automatas.***
Let $\Sigma = \{a, b\}$. Construst a deterministic automata that accepts the following:

1. No word over Σ;
2. Any word over Σ;
3. Only the words composed of one symbol of Σ;
4. $a^* \cup b^*$.

<u>*Answer:*</u>

1.


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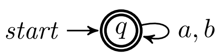

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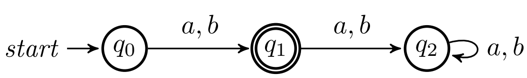

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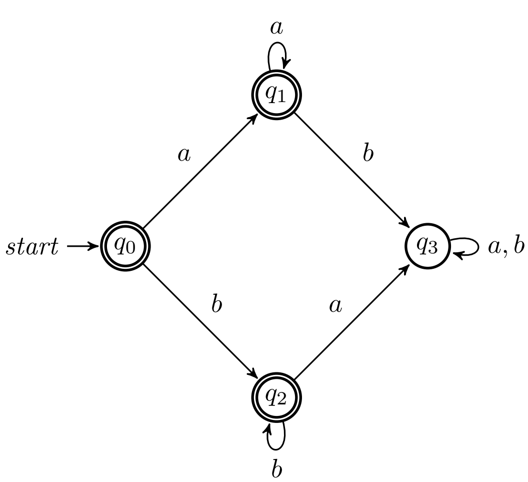

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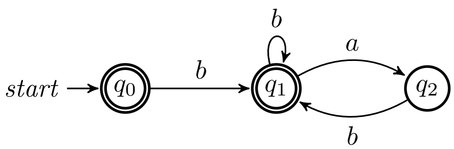
Exercise 2 : ***Salad of automatas.***
Let $\Sigma = \{a, b\}$. Construct an automata that accepts the following languages:

1. $L_1 = \{w \in \Sigma^* \mid \text{every } a \text{ in } w \text{ is immediately preceded and followed by a } b\}$;
2. $L_2 = \{w \in \Sigma^* \mid ab \notin w \text{ and } ba \notin w\}$;
3. $L_3 = \{w \in \Sigma^* \mid ab \in w \text{ or } ba \in w\}$;
4. $L_4 = \{w \in \Sigma^* \mid ab \in w \text{ and } ba \in w\}$;

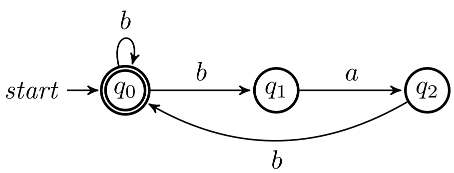
5. $L_5 = \{w \in \Sigma^* \mid w$ contains an odd number of a and an odd number of $b\}$;
6. $L_6 = \{w \in \Sigma^* \mid aaa \notin w$ and w contains an odd number of $b\}$;
7. $L_7 = \{w \in \Sigma^* \mid aa \in w$ and the first occurrence of aa is not preceded by $abab\}$.

Answer:
We will not build complete automatas. If, when reading a word, the automata cannot follow a transition from its current state, we consider that it rejects the word.

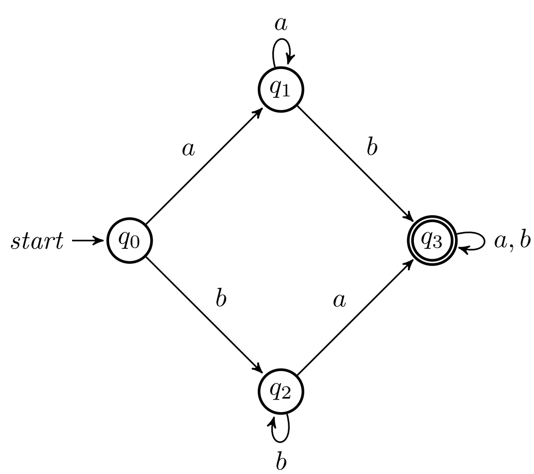
1. *Deterministic solution:*



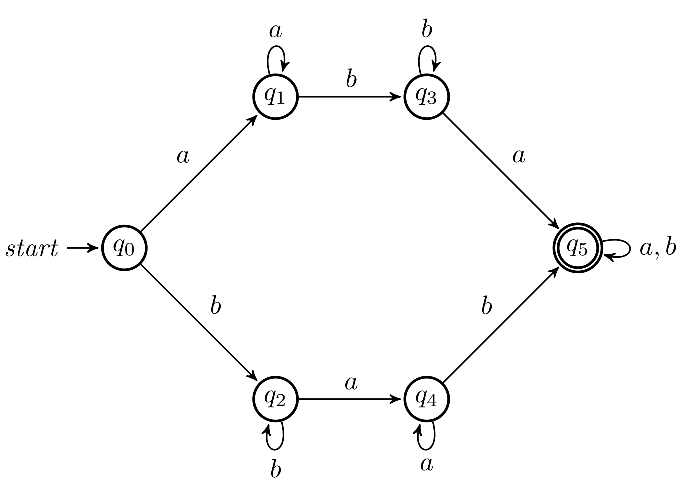
Non-deterministic solution:



2. *This language is equivalent to $a^* \cup b^*$ (so use the automata in Exercise 1).*

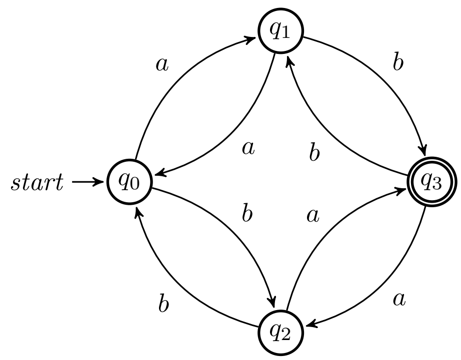


3. *This language is the complement of $a^* \cup b^*$.*

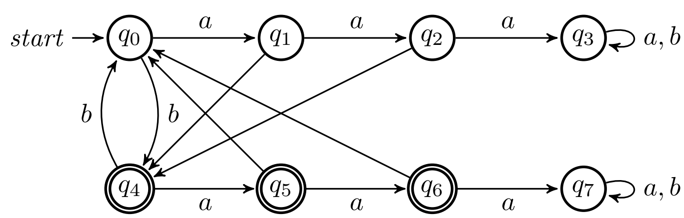


4.

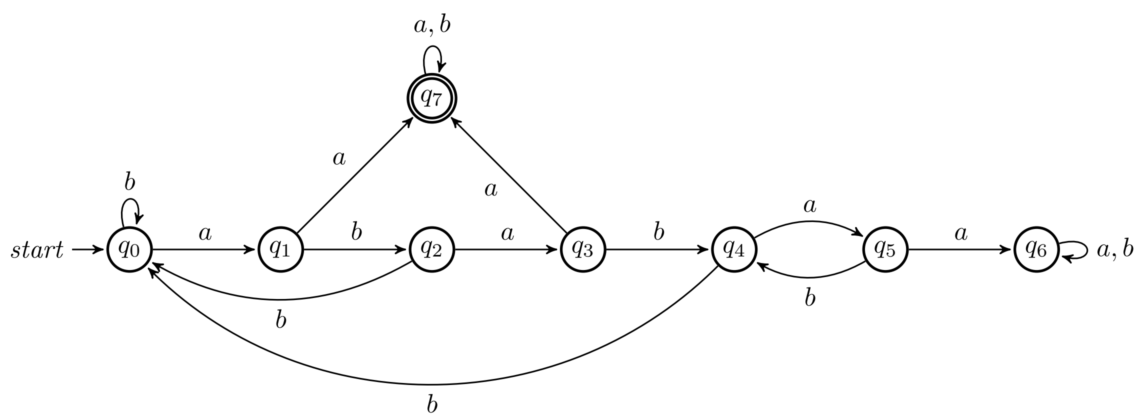
5. *In the following automata, q_0 is the state where there is an even number of both a and b, q_1 is the state where there is an odd number of a and an even number of b, q_2 is the state where there is an even number of a and an odd number of b, and q_3 is the state where there is an odd number of both a and b:*



6. *In the following automata, the top row is when we have read an even number of b, the bottom row when we have read an odd number of b. Each column corresponds to, respectively, 0, 1, 2 and 3 consecutive a. The back edges from q_1, q_2, q_5 and q_6 are all labeled with b.*



7. *In the following automata, the state q_7 is reached when the first aa has been read and has not been preceded by $abab$, leading to us accepting the word. If the first aa is preceded by $abab$, then we reach the state q_6, and will not accept the word.*



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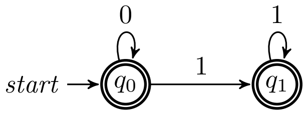
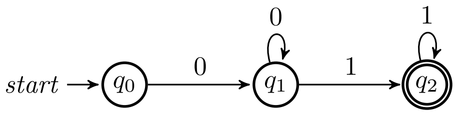
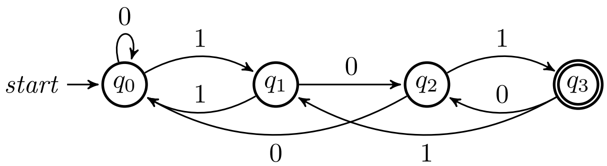
Exercise 3 : ***Binary words.***
Let $\Sigma = \{0, 1\}$. Construct an automata that accepts the following languages:

1. $L_0 = \{0^m 1^n \mid m, n \geq 0\}$;
2. $L_1 = \{0^m 1^n \mid m, n > 0\}$;
3. $L_{10} = \{w \in \Sigma^* \mid w \text{ ends with } 101\}$;
4. $L_{11} = \{w \in \Sigma^* \mid w \equiv 0 \bmod 3\}$.

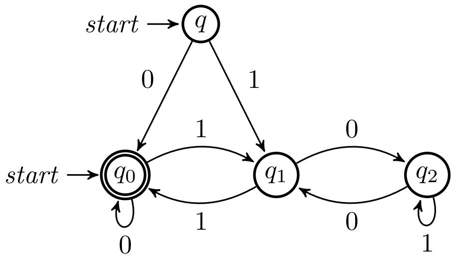
Answer:

1. 

2. 

3. 

4. *We will use four states. The state q is the initial state, and serves to verify that we do not read the empty word. The state q_i will be reached if the partial word being read is congruent to i modulo 3. When reading a binary number a, we can deduce the class of congruence modulo 3 of $a{\cdot}0$ and $a{\cdot}1$ depending on the class of a. For example, if $a = 3k$, then $a{\cdot}0 = 6k$ and $a{\cdot}1 = 6k + 1$. The other two cases work similarly, and we can deduce the following automata:*



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Exercise 4 : *Binary operations*.
Let L be a language. We define L^R as the language containing all the words in L written in reverse (for example, if L contains *abab* and *xkcd* then L^R contrains *baba* and *dckx*).

1. Prove that if L is regular, then L^R is regular.

Let $\Sigma_2 = \left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$. Note that any word $w \in \Sigma_2^+$ can be read as two binary numbers, one for each row. The top number is denoted by w^t and the bottom number by w^b.

2. Prove that the language $L = \{w \in \Sigma_2^+ \mid w^t > w^b\}$ is regular.

Let $\Sigma_3 = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \dots, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$. Again, for every word $w \in \Sigma_3^+$ we get three binary numbers. The top number is w^t, the middle number is w^m and the bottom number is w^b.

3. Prove that the language $L = \{w \in \Sigma_3^+ \mid w^b = w^t + w^m\}$ is regular.

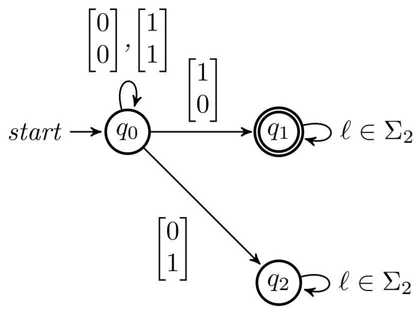
<u>*Answer:*</u>

1. *Let $A = (Q, \Sigma, \delta, q_0, F)$ be a finite automata that recognizes L. We construct the automata $A^R = (Q^R, \Sigma^R, \delta^R, q_0^R, F^R)$ the following way:*
   - $\Sigma^R = \Sigma$;
   - *A^R has all the states of A, that is, $Q^R = Q$;*
   - *For all q, q', ℓ such that $\delta(q, \ell) = q'$, we have $\delta^R(q', \ell) = q$, that is, all the transitions of A are reversed in A^R;*

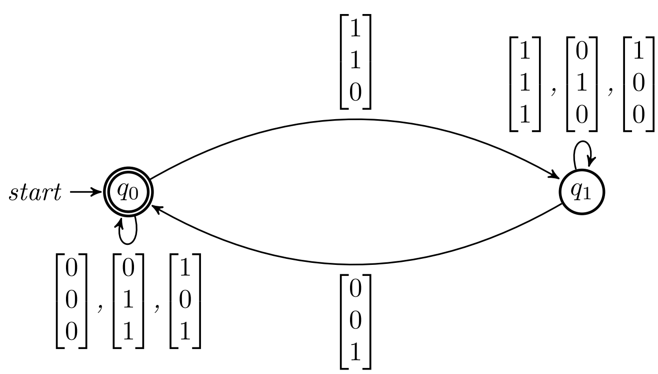
- $F^R = \{q_0\}$, *that is, the starting state of* A *is an accepting state of* A^R;
- *If* A *has only one accepting state* q_f, *then it becomes the starting state of* A^R: $q_0^R = q_f$;
- *If* A *has several accepting states, that is, we have* $F = \{q_{f_1}, \ldots, q_{f_k}\}$, *then we create the starting state* q_0^R *of* A^R, *and add transitions* $\delta^R(q_0^R, \epsilon) = q_{f_i})$ *for all* $i \in \{1, \ldots, k\}$, *where* ϵ *is the empty symbol.*

It is easy to see that A^R *recognizes* L^R. *Since* A^R *is a finite automata, this implies that* L^R *is regular (note that* A^R *may not be deterministic, but this is not a problem since deterministic and non-deterministic automatas recognize the same class of languages).*

2. *We construct a finite automata that recognizes* L. *In* q_0, *the two numbers are still equal. We enter* q_1 *when the top number is bigger than the bottom number, and* q_2 *in the opposite case.*



3. *We construct an automata that recognizes* L^R, *which will prove by the point above that* L *is regular. For this, we simply do a binary addition with carry. In state* q_0, *there is no carry, and in state* q_1, *there is a carry. Recall that a word is rejected if it is in a state where its character being read does not have a transition arc labelled with this character.*



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Exercise 5 : ***The Baker Street Irregulars.***
Use the pumping lemma to prove that the following languages are not regular:

1. $L_1 = \{a^n b^n \mid n \geq 0\}$;
2. $L_2 = \{a^{n^2} \mid n > 0\}$;
3. $L_3 = \{a^n b^{2n} \mid n > 0\}$;
4. $L_4 = \{(ab)^n c^n \mid n \geq 0\}$.

Use the properties of regular languages to prove that the following language is not regular:

5. $L_5 = \{a^m b^n \mid m \neq n\}$.

Prove in two different ways that the following language is not regular:

6. $L_6 = \{w \in \{a, b\}^* \mid w = w^R\}$ (that is, L is the language of all the palindromes over $\{a, b\}$).

Hint: One method uses the pumping lemma, the other one the fact that an automata accepting L is finite.
Answer: First we recall the pumping lemma:
If L is regular, then there exists an integer p such that, for all $w \in L$ with $|w| \geq p$, we can write $w = xyz$ where $|y| > 0$, $|xy| \leq p$ and $\forall n \geq 0$ $xy^n z \in L$.

1. *Assume that L_1 is regular, we apply the pumping lemma. Let $w = a^p b^p = xyz$, and since $|xy| \leq p$ we have $xy = a^{|xy|}$. In particular, let $k = |y|$. Then, we know that for any integer n, $xy^n z = a^{p+(n-1)k} b^p \in L$. However, this implies that $p + (n-1)k = p$, which in turn implies $k = 0$, a contradiction since $|y| > 0$.*

2. *Assume that L_2 is regular, we apply the pumping lemma. Let $w = a^{p^2} = xyz$. Note that the next word in L is $w_+ = a^{(p+1)^2} = a^{p^2+2p+1}$. Furthermore, since $|xy| \leq p$, we have $|y| \leq p$. Assume that $|y| = p$, the highest possible value. But then $xy^2 z = a^{p^2+p} \notin L$ (since it is longer than w and shorter than w_+). All the other possible values for $|y|$ yield the same issue, leading to a contradiction.*

3. *Assume that L_3 is regular, we apply the pumping lemma. Let $w = a^p b^p = xyz$. We have $|y| = k > 0$, and as such $xy^0 z = xz = a^{p-k} b^p \notin L$, a contradiction.*

4. *Assume that L_4 is regular, we apply the pumping lemma. Let $w = (ab)^p c^p = xyz$. There are four possible values for y: $y_1 = (ab)^k$, $y_2 = (ba)^k$, $y_3 = b(ab)^k$ and $y_4 = a(ba)^k$. Now we only need to prove that there exists an integer n such that $xy^n z \notin L$ in each of those four cases, which will lead to a contradiction. For instance, $xy_1^2 z = (ab)^{p-k}(ab)^{2k} c^p \notin L$ since $p - k + 2k = k \Rightarrow k = 0$ which is impossible.*

5. *Assume that L_5 is regular. Let $\overline{L_5} = \{a, b\}^* \setminus L$. Since the regular languages are closed under complementation, $\overline{L_5}$ is regular. Furthermore, let $M = a^* b^*$. It is easy to see that M is regular. Now, the regular languages are closed under intersection, and thus $\overline{L_5} \cap M$ is regular. However, $\overline{L_5} \cap M = L_1$, which is not regular as we saw previously. This contradiction proves that L_5 is not regular.*

6.1. *Using the pumping lemma. Assume that L_6 is regular. Let $w = a^p b a^p = xyz$. Then, we have $y = a^k$ and as such $xy^2 z = a^{p+k} b a^p \notin L$ since $k \neq 0$, which is a contradiction.*

6.2 *Assume by contradiction that L_6 is regular, then there exists a finite automata A that accepts every word of L_6. Let us consider $w = a^n b a^n$. When reading the word w, A will reach the state q_n after having read $a^n b$. Since A is finite, there exists an integer $m \neq n$ such that $q_m = q_n$, that is, A reaches the state q_n after having read $a^m b$. However, after finishing reading w, A will have reached a terminating state since $w \in L$. But this implies that when A reads the word $w' = a^m b a^n$, it reaches a terminating state and as such accepts the word w'. This is a contradiction, since w' is not a palindrome over $\{a, b\}$ since $m \neq n$.*

$\square$