Computability and Complexity

Homework

Finite automatas and regular languages

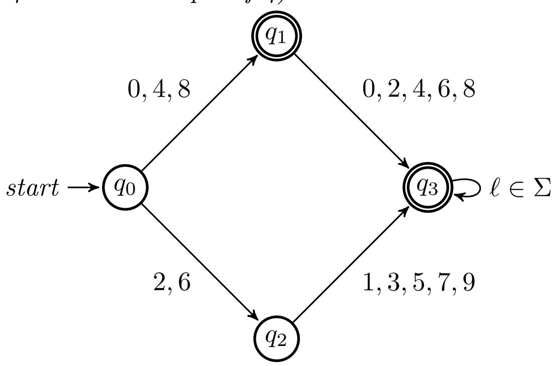
Exercise 1 : ***Reverting.***
Prove that if L is a regular language, then L^R (the language consisting in all the words in L in reverse) is regular.
You may reuse this result in some of the following exercises.
Answer: See the corresponding exercise sheet.

□

Exercise 2 : ***Multiples of 4.***
Let $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (thus Σ^+ is the set of nonnegatives integers written in decimal). Prove that $L = \{w \in \Sigma^* \mid w \equiv 0 \bmod 4\}$ (the language of the multiples of 4) is regular.
Answer: We know that $n \equiv 0 \bmod 4 \Leftrightarrow$ the last two digits of n are a multiple of 4. Thus, we need to check if the last two digits of w are a multiple of 4. The easiest way to do it is to study L^R. Now there are a few cases:

- *If w is empty, then it is rejected. If w is one character, it is accepted if and only if $w \in \{0, 4, 8\}$.*
- *If w ends with an odd digit, then it is rejected.*
- *If w ends with a 0, then it has to be preceded by an even digit.*
- *If w ends with a 2, then it has to be preceded by an odd digit.*
- *It is easy to notice that ending with a 4 or a 8 is equivalent to ending with a 0, and that ending with a 6 is equivalent to ending with a 2.*

Hence we construct an automata that accepts L^R, proving that L is regular (note that if you read the word but cannot follow a transition, then the word is rejected; for example the word 41 is rejected since 14 is not a multiple of 4):



An alternative would be to construst a nondeterministic finite automata to check if the last two digits of w are a multiple of 4.

□

Exercise 3 : ***Intersectionality.***
Prove that the regular languages are closed under intersection (*i.e.*, if L_1 and L_2 are regular languages, then $L_1 \cap L_2$ is a regular language).

Answer: We can assume that the two languages share the same alphabet Σ. Let $M_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_0^2, F_2)$ be two finite automatas that recognize L_1 and L_2, respectively. We construct an automata $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $L_1 \cap L_2$:

- $Q = Q_1 \times Q_2$;
- $q_0 = (q_0^1, q_0^2)$;
- $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ and } r_2 \in F_2\}$;
- $\delta((r_1, r_2), \ell) = (\delta_1(r_1, \ell), \delta_2(r_2, \ell))$ *for all* $\ell \in \Sigma$.

It is easy to check that a word is accepted by M if and only if it is in both L_1 and L_2.

□

Exercise 4 : *Comparing sums.*

Let $\Sigma_4 = \left\{ \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \ldots, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$. Thus, any word w on Σ_4^* induces four binary numbers, called w_1, w_2, w_3, w_4 (from top to bottom).

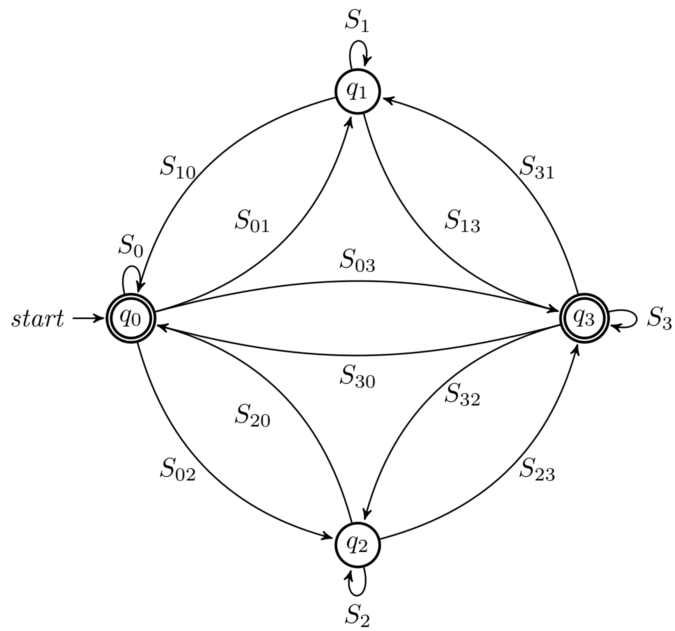
Prove that $L = \{w \in \Sigma_4^* \mid w_1 + w_2 = w_3 + w_4\}$ is regular.

Answer: We recognize L^R. There are four states, depending on whether there is a carry in the top sum (q_1) or in the bottom sum (q_2) or in both (q_3) or in neither (q_4). We check, at each turn, whether $w_1^i + w_2^i$ (plus the eventual top sum carry) is equal to $w_3^i + w_4^i$ (plus the eventual bottom sum carry). The word is accepted if both sums are in the same carry state at the end. Note that if you read the word but cannot follow a transition, then the word is rejected. We define the following sets:

- $S_0 = \left\{ \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \right\}$ *(there was no carry and there is still no carry);*
- $S_{01} = \left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right\}$ *(the top row gains a carry);*
- $S_{10} = \left\{ \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$ *(the top row loses its carry);*
- $S_1 = \left\{ \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \right\}$ *(the top row retains its carry, the bottom row still does not have a carry);*
- $S_{13} = \left\{ \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix} \right\}$ *(the top row retains its carry, and the bottom row gains a carry);*
- $S_{31} = \left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right\}$ *(the top row retains its carry, and the bottom row loses its carry);*

- $S_3 = \left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \right\}$ *(both have carry and retain their carry);*
- $S_{03} = \left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$ *(both rows gain a carry);*
- $S_{30} = \left\{ \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \right\}$ *(both rows lose their carry);*
- *The sets* $S_{02}, S_{20}, S_2, S_{23}$ *and* S_{32} *are the symmetrics of sets* $S_{01}, S_{10}, S_1, S_{13}$ *and* S_{31} *respectively (concerning the bottom row carry);*

and get the following automata:



□

Exercise 5 : ***One state to accept them all.***
Prove that every nondeterministic finite automata can be converted to an equivalent one that has a single accept state.
Answer: Let $M = (Q, \Sigma, \delta, q_0, F)$ *be a nondeterministic finite automata. We construct* $M' = (Q', \Sigma', \delta', q'_0, F')$ *such that* $|F'| = 1$ *and* M' *recognizes the same language than* M*:*

- $Q' = Q \cup \{q_f\}$;
- $\Sigma' = \Sigma$;
- $q'_0 = q_0$;
- $F' = q_f$;
- *For all* $q \in Q$ *and* $\ell \in \Sigma \cup \{\epsilon\}$, $\delta'(q, \ell) = \delta(q, \ell)$;
- *For all* $q \in F$, $\delta'(q, \epsilon) = q_f$.

It is easy to see that if M *accepts a word, then* M' *will accept it too, and conversely.*

□

Exercise 6 : *Pumping.*

Use the pumping lemma to prove that $L = \{a^m b^n \mid m > n\}$ is not regular.

Answer: Assume that L is regular, we apply the pumping lemma. Let $w = a^{p+1}b^p = xyz$. Since $|xy| \leq p$ and $|y| \geq 1$, $y = a^k$. Then, $xy^0z = xz = a^{p+1-k}b^p \notin L$, a contradiction. Thus, L is not regular.

□

Exercise 7 : *Huey, Dewey and Louie.*

Let $\Sigma_2 = \left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$. Thus, any word w on Σ_2^* induces two binary numbers, called w_1 and w_2 (from top to bottom).

1. Prove that $L = \{w \in \Sigma_2^* \mid 3w_1 = w_2\}$ (*i.e.*, where the bottom row is three times the top row) is regular.
2. Prove that $L = \{w \in \Sigma_2^* \mid w_1 = w_2^R\}$ (*i.e.*, where the bottom row is the reverse of the top row) is not regular.

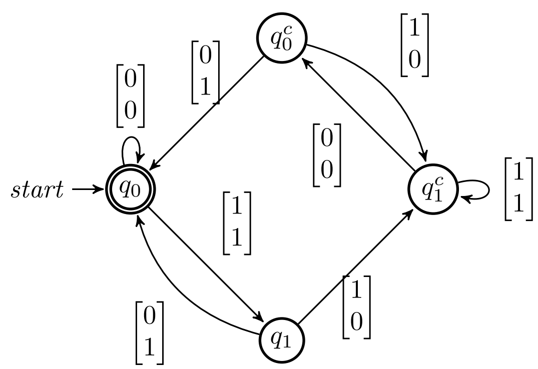
Answer:

1. *Let $w_1 = a_n 2^n + a_{n-1}2^{n-1} + \ldots + a_1 2 + a_0$. Now:*

$$3w_1 = (2+1)w_1 = (a_n 2^{n+1} + a_{n-1}2^n + \ldots + a_0 2 + 0) + (a_n 2^n + a_{n-1}2^{n-1} + \ldots + a_1 2 + a_0)$$

and we can express $w_2 = 3w_1$ as: $a_0 = b_0$ and $a_i + a_{i-1} = b_i$ (and we need to take the carry into account). Thus we need to check those two conditions, which can be done by recognizing L^R and having states for remembering the value of a_{i-1} and the fact that we have a carry or not. We can only accept the word if there is no carry at the end and if $a_n = 0$ (since otherwise we would need another digit in w_2).

We construct the following automata, where q_j is a state where $a_{i-1} = j$ and q_j^c denotes that there is a carry (recall that if you read the word but cannot follow a transition, then the word is rejected):



We can check the correctness. For example: if we are in q_0, then $a_{i-1} = 0$ and there is no carry, and thus when reading $\begin{bmatrix}0\\0\end{bmatrix}$ we have $a_i + a_{i-1} = 0 = b_i$ and we remain in the same state since $a_i = 0$. When reading $\begin{bmatrix}1\\1\end{bmatrix}$, $a_i + a_{i-1} = 1 = b_i$ and we go to q_1 since $a_i = 1$. Reading any of the other two characters breaks the condition, and thus we reject the word. The other transitions can be verified similarly.

2. *Assume by contradiction that L is regular, we apply the pumping lemma. Let $w = \begin{bmatrix}0\\1\end{bmatrix}^p \begin{bmatrix}1\\0\end{bmatrix}^p = xyz$. Since $|xy| \leq p$ and $|y| \geq 1$, we have $y = \begin{bmatrix}0\\1\end{bmatrix}^k$, and thus $w' = xy^2z = \begin{bmatrix}0\\1\end{bmatrix}^{p+k} \begin{bmatrix}1\\0\end{bmatrix}^p$. However, in w', we have $w_1 = 0^{p+k}1^p$ and $w_2 = 1^{p+k}0^p$, and thus $w' \notin L$, a contradiction.*

□