Computability and Complexity

Homework

Turing machines, decidability and complexity

Exercise 1 : *The complexity of graph colouring*.
Let $G(V, E)$ be a graph with vertex set V and edge set E. A k-colouring of the graph is a function $\alpha : V \to \{1, \dots, k\}$ such that, for any two vertices u and v, if $uv \in E$ then $\alpha(u) \neq \alpha(v)$. In natural language, this means that we associate integers to vertices, and that two adjacent vertices cannot have the same associated integer.
The goal of this exercise is to study the complexity of deciding if, for a given graph G and a given integer k, there exists a k-colouring of G. Such a problem is called k-COLOR, and it is formally defined as follows:

$$k\text{-COLOR} = \{< G, k > \mid G \text{ is an undirected graph that has a } k\text{-colouring}\}.$$

This is a classical problem in complexity theory, and we already proved in the class that 3-COLOR is NP-complete. We will now study its complexity for other values of k, as well as see if restricting the class of G can change the complexity.

1. Prove that 1-COLOR is in P.
2. Prove that 2-COLOR is in P. Hint: What graphs are 2-colourable?
3. Let $k \geq 3$. Prove that k-COLOR is in NP.
4. Find a polynomial transformation that, from any graph G, constructs a graph G' such that G is 3-colourable if and only if G' is 4-colourable. Make sure to prove that the transformation is polynomial and correct, and draw an example.
5. Deduce from the previous two questions the complexity class of 4-COLOR.
6. Generalize your construction from question 4 to any integer $k > 3$. Deduce from this the complexity class of k-COLOR.

We are now studying 3-COLOR. We know that this problem is NP-complete when G is a general graph. However, several NP-complete problems become polynomial when restricted to particular graph classes. An interesting class is the class of *planar graphs*. Formally, planar graphs are the class of $(K_5, K_{3,3})$-minor-free graphs. Informally (but usefully!), they are the graphs that we can draw without any edge crossing another. Such graphs are interesting since their structure is well constrained, and several NP-complete problems have polynomial algorithms for planar graphs, such as MAX-CUT or GRAPH ISOMORPHISM. We will study the complexity of 3-COLOR restricted to planar graphs.

7. Consider the graph H depicted on Figure 1. Prove the following properties of H:
   (a) For any 3-colouring α of H, we have $\alpha(x) = \alpha(x')$ and $\alpha(y) = \alpha(y')$.
   (b) There exist two 3-colourings α_1 and α_2 of H such that $\alpha_1(x) = \alpha_1(y)$ and $\alpha_2(x) \neq \alpha_2(y)$.
8. Use H to polynomially construct, for any graph G, a planar graph G' such that G has a 3-colouring if and only if G' has a 3-colouring.
9. Deduce from the previous question the complexity class of 3-COLOR restricted to planar graphs.

Answer:

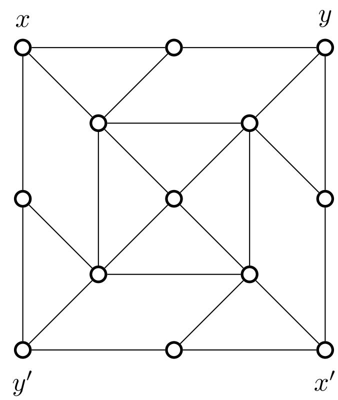

Figure 1: A graph H with an interesting property.

1. *A graph is 1-colourable if and only if it is an independent set. Thus, there is a polynomial algorithm solving* 1-COLOR*: simply test whether there is an edge in the graph (and answer No) or not (and answer Yes).*

2. *A graph is 2-colourable if and only if it is bipartite. Thus, answering* 2-COLOR *on G is exactly equivalent to deciding whether G is bipartite. An algorithm answering this is a BFS of the graph, which is polynomial. Thus, there is a polynomial algorithm solving* 2-COLOR*: apply a BFS and greedily colour every vertex encountered. If, at the end, more than two colours were used, answer No; otherwise, answer Yes.*

3. *Let C be a certificate for k-*COLOR*. Namely, C is a k-colouring, that is, a function assigning a colour to every vertex. Here is an algorithm testing whether this certificate is valid: simply verify that, for every colour i, no two vertices coloured with i are adjacent. This algorithm is clearly polynomial (it executes in $O(k|V|^2)$ steps), which implies that k-*COLOR *is in NP.*

4. *Let $G(V, E)$ be a graph. We construct $G'(V', E')$ the following way:*

   - $V' = V \cup \{s\}$
   - $E' = E \cup \{su \mid u \in V\}$

   This transformation is polynomial, since $|V'| = |V| + 1$ and $|E'| = |E| + |V|$.

   Assume that G has a 3-colouring α. It is easy to construct a 4-colouring α' of G': $\alpha'(u) = \alpha(u)$ for $u \in V$ and $\alpha'(s) = 4$. Conversely, assume that G' has a 4-colouring α'. Then, by our construction, $\alpha'(s) \neq \alpha'(u)$ for every $u \in V$. Thus, by renaming the colours if necessary, we can have $\alpha'(s) = 4$, and now by having $\alpha(u) = \alpha'(u)$ we obtain a 3-colouring α of G.

5. *The transformation described in the previous question is a polynomial reduction from* 3-COLOR *to* 4-COLOR*. Since we also proved that* 4-COLOR *is in NP, and that* 3-COLOR *is NP-complete, this implies that* 4-COLOR *is NP-complete.*

6. *It is easy to generalize the construction for any $k > 3$:*

   - $V' = V \cup \{s_1, \ldots, s_{k-3}\}$
   - $E' = E \cup \{us_i \mid u \in V, i \in \{1, \ldots, k-3\}\} \cup \{s_i s_j \mid i \neq j, i, j \in \{1, \ldots, k-3\}\}$

   *In other words, we create a clique of $k-3$ vertices that are linked to every other vertex in V. The graph we obtain has a k-colouring if and only if G has a 3-colouring. Going from the 3-colouring to the k-colouring is easy (simply assign colours $4, \ldots, k$ to the s_i); and conversely all the s_i have a different colour (since they are a clique) and have a different colour from all vertices in V, so by renaming the colours if necessary we can have the colours 1, 2 and 3 to vertices in V which gives us a 3-colouring. This reduction, combined with the fact that k-*COLOR *is in NP (trivial), proves that the problem is NP-complete.*

7. *This is a case analysis. I am not doing it. Just try all possible 3-colourings (taking symmetry into account, there are only a few cases to consider), it works.*

8. *This construction is quite complicated to write, but quite easy to understand. The idea is to start from a graph G and obtain a planar graph G' such that G is 3-colourable if and only if G' is 3-colourable. The construction is the following:*

   - *Place all the vertices from G, say $v_1, \ldots, v_n$, in line.*
   - *Draw every edge $v_i v_{i+1}$.*
   - *For all other edges $v_i v_j$ (with $i < j$), do the following:*
     - *Use H to "invert" the positions of v_i and v_{i+1} (associate them with x and y, and their inverted equivalents one level down will be x' and y'). For all other vertices, use a basic template to get them down one level too.*
     - *Iterate until v_i and v_j are neighbours in the line. Link them, then return v_i to its initial position by using the same process.*

   This is depicted on Figure 2. Now, it is easy to see that this transformation is polynomial: we add at most $|V|$ such "intermediate levels" for every edge, and each level contains one copy of H (9 new vertices and 24 new edges) and $|V| - 2$ basic templates (2 new vertices and 5 new edges for each); plus the levels with the v_i^j (which are obviously polynomial in size). Thus, every intermediate level is of size $\theta(|V|)$, and we add at most $|V||E|$ of those levels. So the graph G' that we obtain is of polynomial size compared to the original graph G. Furthermore, it is planar, since no two edges cross in our construction.

   Now, assume that G has a 3-colouring α. It is easy to see that we can construct a 3-colouring α' of G': for every i and j we have $\alpha'(v_i^j) = \alpha(v_i)$ (which is always possible due to the properties of H and of the basic template), and we colour every vertex in the copies of H and of the basic template any way that is possible. We will not have a problem since $v_i^k v_j^k \in E'$ if and only if $v_i v_j \in E$.

   Conversely, assume that G' has a 3-colouring α'. It is easy to see that if v_i^j and v_i^{j+1} are linked by a basic template, then $\alpha'(v_i^j) = \alpha'(v_i^{j+1})$. Furthermore, if $v_i^j, v_{i+1}^j, v_i^{j+1}$ and v_{i+1}^{j+1} are linked through a copy of H, then they are respectively x, y, x' and y'. By the properties of H, we have $\alpha'(v_i^j) = \alpha'(v_i^{j+1})$ and $\alpha'(v_{i+1}^j) = \alpha'(v_{i+1}^{j+1})$, and they can be indifferently of the same colours or of different colours. Thus, for every i and j, we have $\alpha'(v_i^j) = \alpha'(v_i^0)$ where the v_i^0's are the initial row of vertices. Thus, we can construct a 3-colouring α of G by having $\alpha(v_i) = \alpha'(v_i^0)$. Again, we will not have any problem since $v_i^k v_j^k \in E'$ if and only if $v_i v_j \in E$.

9. *Since* 3-COLOR *is in NP, its restriction to planar graphs is in NP too. The previous question gave us a polynomial reduction from* 3-COLOR *to* 3-COLOR *restricted to planar graphs. We know that* 3-COLOR *is NP-complete, and thus* 3-COLOR *restricted to planar graphs is NP-complete too.*

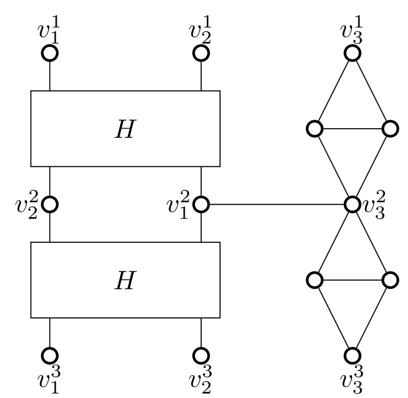

Figure 2: An example of the reduction from 3-COLOR to PLANAR-3-COLOR. Here there is an edge $v_1 v_3$, so we have to use H to get v_1 next to v_3.

□