Instituto de Matemáticas

Homework

Turing machines, decidability and complexity

Exercise 1 : ***The complexity of graph colouring*.**
Let $G(V, E)$ be a graph with vertex set V and edge set E. A k-colouring of the graph is a function $\alpha : V \to \{1, \ldots, k\}$ such that, for any two vertices u and v, if $uv \in E$ then $\alpha(u) \neq \alpha(v)$. In natural language, this means that we associate integers to vertices, and that two adjacent vertices cannot have the same associated integer.
The goal of this exercise is to study the complexity of deciding if, for a given graph G and a given integer k, there exists a k-colouring of G. Such a problem is called k-COLOR, and it is formally defined as follows:

$$k\text{-COLOR} = \{< G, k > \mid G \text{ is an undirected graph that has a } k\text{-colouring}\}.$$

This is a classical problem in complexity theory, and we already proved in the class that 3-COLOR is NP-complete. We will now study its complexity for other values of k, as well as see if restricting the class of G can change the complexity.

1. Prove that 1-COLOR is in P.
2. Prove that 2-COLOR is in P. Hint: What graphs are 2-colourable?
3. Let $k \geq 3$. Prove that k-COLOR is in NP.
4. Find a polynomial transformation that, from any graph G, constructs a graph G' such that G is 3-colourable if and only if G' is 4-colourable. Make sure to prove that the transformation is polynomial and correct, and draw an example.
5. Deduce from the previous two questions the complexity class of 4-COLOR.
6. Generalize your construction from question 4 to any integer $k > 3$. Deduce from this the complexity class of k-COLOR.

We are now studying 3-COLOR. We know that this problem is NP-complete when G is a general graph. However, several NP-complete problems become polynomial when restricted to particular graph classes. An interesting class is the class of *planar graphs*. Formally, planar graphs are the class of $(K_5, K_{3,3})$-minor-free graphs. Informally (but usefully!), they are the graphs that we can draw without any edge crossing another. Such graphs are interesting since their structure is well constrained, and several NP-complete problems have polynomial algorithms for planar graphs, such as MAX-CUT or GRAPH ISOMORPHISM. We will study the complexity of 3-COLOR restricted to planar graphs.

7. Consider the graph H depicted on Figure 1. Prove the following properties of H:
   (a) For any 3-colouring α of H, we have $\alpha(x) = \alpha(x')$ and $\alpha(y) = \alpha(y')$.
   (b) There exist two 3-colourings α_1 and α_2 of H such that $\alpha_1(x) = \alpha_1(y)$ and $\alpha_2(x) \neq \alpha_2(y)$.
8. Use H to polynomially construct, for any graph G, a planar graph G' such that G has a 3-colouring if and only if G' has a 3-colouring.
9. Deduce from the previous question the complexity class of 3-COLOR restricted to planar graphs.

□

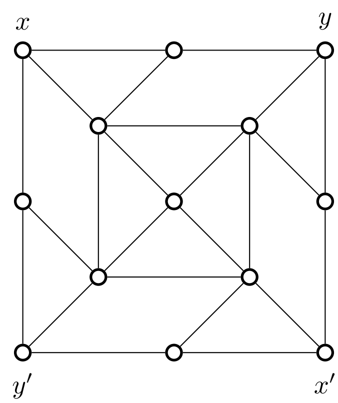

Figure 1: A graph H with an interesting property.