# Internship Report:
# Octal Games on Graphs

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**Abstract.** Combinatorial games are two-player games with perfect information, without chance, loops or draws, and where the last move entirely determines the winner. Octal games are combinatorial games played on heaps of counters whose rules are entirely determined by an octal code. During my internship, I defined an extension of octal games to graphs and studied two of these games on trees and grids, using a program to help conjecture a result before formally proving it.

**Résumé** Les jeux combinatoires sont des jeux opposant deux joueurs, à information parfaite, sans hasard, boucles ou match nul, et où le dernier coup joué détermine entièrement le vainqueur. Les jeux octaux sont une famille de jeux combinatoires se jouant sur des piles d'objets, et dont les règles sont entièrement déterminés par un code octal. Durant mon stage, j'ai défini une extension des jeux octaux sur les graphes et étudié deux de ces jeux sur les arbres et les grilles, utilisant un programme pour conjecturer un résultat avant de le démontrer formellement.

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## Introduction

Games are a huge part of human history. However, formal mathematical study of games is a relatively recent field. Use of the probability theory for the study of gambling games has arisen during the XVI-XVIIth century, and has recieved a lot of attention in the XXth century, with the work of Cournot, Borel, von Neumann, Nash... and the other researchers who developed the game theory.

Some games lend themselves to a study over a more particular theory. Two-player games with perfect information, without chance, loops and draws, and where the last move entirely determines the winner, are called combinatorial games. Two winning conventions are defined: under the *normal* convention, the last player to play a move wins; while under the *misère* convention, the last player to play a move loses. During the XXth century, a formal theory has arisen to study these games, with powerful mathematical tools.

The first combinatorial game that has been studied is the game of NIM, in 1901 [Bouton, 1901]. In a game of NIM, there are several heaps of counters. The two players must alternatively remove counters from exactly one heap, the winner under the *normal* convention being the player who removes the last counter from the last heap.

During the next few years, several games were studied by mathematicians. Most of these games were specific in that both players always had the same moves available to them (as opposed to games like chess). Such games are called impartial games.

In 1935 [Sprague, 1935] and 1939 [Grundy, 1939], Sprague and Grundy independently proved that every impartial game is equivalent to a certain position of NIM. This gave way to a more systematic study of games, by giving them numerical values which are called Grundy values, followed by the development of formal tools by mathematicians such as Conway [Conway, 1976].

Octal games are a family of impartial combinatorial games, which are represented by an octal code. Octal games are played on heaps of counters, like NIM, and their code entirely determines how many and in which conditions the counters can be taken. Many octal games have been studied and solved, even though some of them are still open [Berlekamp et al., 2001].

Combinatorial games are a formal tool which can have various applications, such as network security or resource allocation. Besides, it is a vast field of research with many open problems, in correlation with other domains such as graph theory or algebra.

This internship, which took place in the GOAL (Graphes, algOrithmes et AppLications) team, in the LIRIS laboratory at Université Claude Bernard Lyon 1, extends the concept of octal games from heaps to the more complex structure of graphs. Indeed, as more and more octal games are being solved, leading to the conjecture that the sequence of the Grundy values of any octal game is eventually periodic, it becomes interesting to examine how these games behave on more complex structures. Graphs are a natural candidate for extending the definition of octal games.

In the first section of this report, we will introduce the concept of combinatorial games and, through illustrated examples, define some of the tools which have been used during the internship. The second section will conclude the state of the art by defining and illustrating octal games.

The third and fourth sections will form the bulk of the report, being a summary of the contributions this internship brought to the domain. The third section will explain results for the 0.03 game, which consists in taking an edge from the graph without disconnecting it, on various classes of graphs, such as forests and grids. We will also detail an algorithm we implemented to help us to formulate the result on $3 \times n$ grids, and which makes us state an interesting conjecture. The fourth section will focus on the 0.33 game, which consists in taking a vertex or an edge from the graph without disconnecting it, on some classes of trees.

# 1 Combinatorial Games

## 1.1 Definitions

Combinatorial games constitute a class of games with a strict definition. To be qualified as a combinatorial game, a game must meet the following conditions:

**Definition 1** *[Albert et al., 2007] A game is a combinatorial game if its rules meet the following conditions:*

1. *There are exactly two players, usually called Left and Right, who alternate moves;*
2. *There is no chance involved;*
3. *There is perfect information: both players know exactly which moves are available to them and to their opponent, and the history of moves played so far;*
4. *The game must end;*
5. *The last move entirely determines who the winner is, and there can be no draw.*

Most of the common board and card games are excluded of the field of combinatorial games. For example, the first condition excludes any game requiring more (resp. less) than two players, such as French tarot (resp. such as solitaire); as well as any game where a player can play multiple times in a turn. The second condition excludes any game with dice, such as Monopoly™; as well as any game with shuffling cards, such as bridge. The third condition excludes any game with imperfect information, such as poker. The fourth condition will exclude any game where players can engage in loops, such as chess. The fifth condition excludes any game where the winner is determined with a score, such as go; as well as any game where draws are possible, such as tic-tac-toe.

However, there are still many games which enter the definition, and more can be created. Moreover, some of the games previously excluded can still be studied thanks to the tools developed for studying combinatorial games! This is the case for the endgames of go, or for some scoring games such as dots&boxes.

The fifth condition tells us that the last move (which will always happen: thanks to the fourth condition, we know that the game will end) will determine the winner. Specifically, there are two different winning conventions:

– The *normal* convention, under which the player who plays the last move wins the game;
– The *misère* convention, under which the player who plays the last move loses the game.

One could think that those two conventions are similar, but they are actually quite different. Most of the tools used to study games under the *normal* convention do not apply under the *misère* convention. Consequently, most of the results in the field of combinatorial games hold under the *normal* convention, although some tools exist to try and tackle some classes of combinatorial games under the *misère* convention.

### 1.2 Impartial games

Let us see an example of combinatorial game: CRAM [Gardner, 1974]. This will allow us to introduce general concepts that will be used later on.

A game of CRAM takes place on a grid, or on part of a grid. Both players will alternate placing dominoes on it, either vertically or horizontally. As we consider the game under the *normal* convention, the first player unable to place a domino loses the game.

Figure 1 shows an example of a game of CRAM. Both players alternate placing dominoes. In this example, the second player wins, as he is the last of the two players to place a domino.

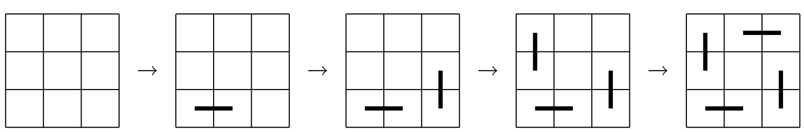

**Fig. 1.** A game of CRAM.

In this report, we will indifferently use the terms *game* and *position* to describe a certain configuration of a game. Consequently, each of the boards from Figure 1 could be called either a game or a position.

**Definition 2** *[Albert et al., 2007] We define the* options *of a game as all the positions that can be reached from it by any player.*

Figure 2 shows an example of a game $G$, along with its options $G_1$, $G_2$, $G_3$ and $G_4$.

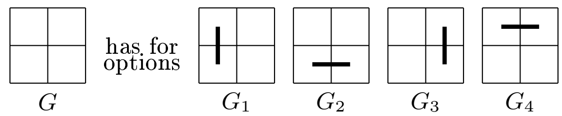


**Fig. 2.** The options of a game of CRAM.

Now, let us consider the grid shown in Figure 3. As we can see, the first player is able to place a domino, resulting in a grid with only one square left. Thus, the second player will be unable to place a domino on it, and will lose the game. So this game is a winning position for the first player, which we call an $\mathcal{N}$-position ($\mathcal{N}$ stands for **N**ext).

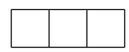

**Fig. 3.** A game of CRAM advantaging the first player, i.e. an $\mathcal{N}$-position.

Now, let us consider the grid shown in Figure 4. We can see that wherever the first player places his domino, the second player will be able to place a second domino. As four squares out of the five will have been taken after those two turns, the first player will be unable to place another domino, and thus he will lose the game. So this game is a winning position for the second player, which we call a $\mathcal{P}$-position ($\mathcal{P}$ stands for **P**revious).

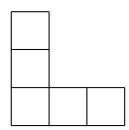

**Fig. 4.** A game of CRAM advantaging the second player, i.e. a $\mathcal{P}$-position.

CRAM is particular in that both players always have exactly the same options. Such games are called impartial games, in opposition to partizan games, where players may have different options available to them.

**Definition 3** *[Albert et al., 2007] A combinatorial game is called an* impartial *game if, for any position of the game, both players have exactly the same options available to them.*

**Definition 4** *[Albert et al., 2007] A combinatorial game which is not an impartial game is called a* partizan *game.*

Partizan games may be games where each player controls his own pieces, such as chess, but there are also partizan versions of impartial games. An example is DOMINEERING, which is a partizan CRAM: as in CRAM, players successively place dominoes on a grid, but unlike CRAM, one of the players can only play horizontal dominoes, while the other can only play vertical dominoes.

As we are not going to study partizan games, we are not going to expand on them.

Instead, we can now introduce the concept of game graph. This is a tool that theoretically allows to study games, but it is not much used in practice due to huge computing costs.

**Definition 5** *[Albert et al., 2007] Given an impartial game $G$, the game graph of $G$ is defined as the directed graph where every vertex is a position reachable from $G$, and where for every vertices $u$ and $v$, the edge $uv$ exists if and only if, $v$ is an option of $u$.*

Figure 5 shows a game graph for the CRAM game. It is constructed inductively from the original game, by playing every available move, and continuing to do so until no moves are left.

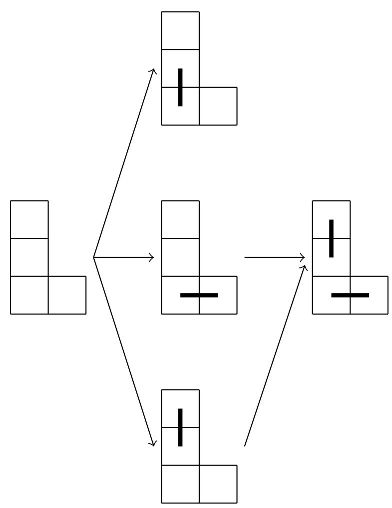

**Fig. 5.** A game graph for a game of CRAM.

We notice that if the first player can play to a $\mathcal{P}$-position, he will ensure that the second player will lose. Otherwise, the first player can only play to $\mathcal{N}$-positions, thus always allowing the second player to win.

In graph theoretical words, the $\mathcal{P}$-positions constitute a kernel of the game graph (i.e. an independent and dominating[1] set).

This gives us an algorithm (Algorithm 1) to determine the outcome of any given game, by using their game graph.

Thanks to the fact that combinatorial games are finite and do not loop, this algorithm ends and will **always** compute the outcome of a game. This can be applied to any impartial game. However, since the size of the game graph can be exponential, it is important to try and solve the games mathematically.

Figure 6 shows the game graph of Figure 5 after applying the algorithm. Thus, we know the original position is an $\mathcal{N}$-position.

The finiteness of this algorithm gives us the following proposition:

[1] A vertex $u$ dominates another vertex $v$ if there is an arc from $v$ to $u$.

**Algorithm 1:** ComputeOutcomeFromGameGraph

```
Data: The game graph G
Result: The outcome of the game
while the root is not marked do
    mark every sink with P
    for every vertex u do
        if a vertex v such that uv is an edge of G is marked P then
            mark u with N
    if the root has been marked then
        return its mark
    delete every marked vertex
```

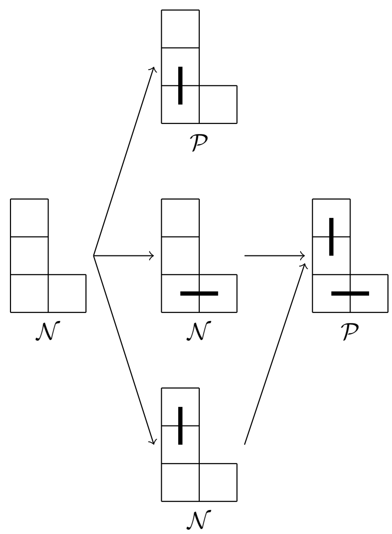


**Fig. 6.** The game graph from Figure 5 marked with Algorithm 1.

**Proposition 6** *[Albert et al., 2007] If, from a position, there exists a move leading to a $\mathcal{P}$-position, then the original position is an $\mathcal{N}$-position.*
*Otherwise, the original position is a $\mathcal{P}$-position.*

Which, in turn, implies that:

**Proposition 7** *[Albert et al., 2007] Any impartial game belongs to exactly one of two outcome classes:*

1. *$\mathcal{N}$ if the first player wins whatever the second player does;*
2. *$\mathcal{P}$ if the second player wins whatever the first player does.*

### 1.3 Sums and values

The game of CRAM has an interesting property: some of its positions can be seen as the conjunction of smaller positions. For example, the game shown at the left of Figure 7 can bee seen as the disjunction of the games shown at the right of the figure.

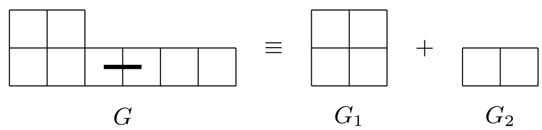


**Fig. 7.** A game of CRAM can be seen as several smaller games of CRAM.

This lets us introducing the concept of disjunctive sum, which will be largely used to study games by breaking large positions into smaller components.

**Definition 8** *[Albert et al., 2007] The disjunctive sum $G + H$ of two games $G$ and $H$ is the game where, at his turn, a player must choose to play one move in exactly one of the games $G$ and $H$. Under the* normal *convention, the winner is the last player able to play a move.*

For example, in the disjunctive sum shown Figure 8, a player can choose to play on either $G$ or $H$.

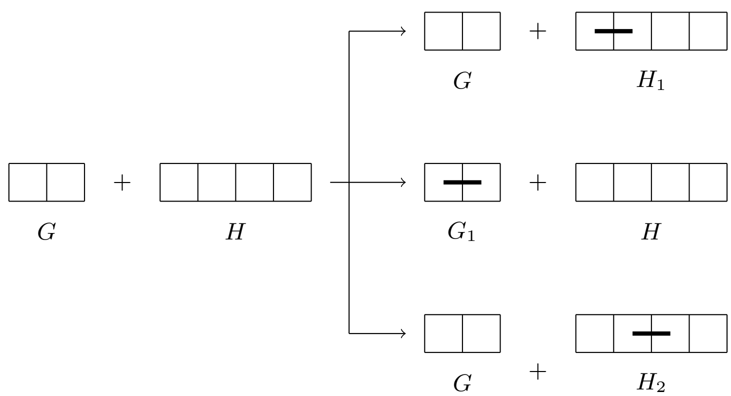


**Fig. 8.** The disjunctive sum of two games of CRAM and its options (minus the symmetries).

**Proposition 9** *[Albert et al., 2007] Let $G$ be a game with an outcome class $O \in \{\mathcal{N};\mathcal{P}\}$. If $H$ is a game with the outcome class $\mathcal{P}$, then $G + H$ has the outcome class $O$.*

This comes from the fact that, when playing on $G + H$, the player with a winning strategy on $G$ will be able to nullify his opponent's moves in $H$: if his opponent plays on $G$, he applies his strategy for this game, otherwise, his opponent plays on $H$, he only has to apply the winning strategy on $H$. As $H$ is a $\mathcal{P}$-position, the player with the winning strategy on $G$ will be the last player to play on it, and thus he will win whether $G$ is already over or not (in the latter case, by applying his strategy on $G$, since his opponent will be forced to play on it).

Thus, the game shown Figure 7 is an $\mathcal{N}$-position: $G_1$ is a $\mathcal{P}$-position, and $G_2$ is an $\mathcal{N}$-position, so $G = G_1 + G_2$ is an $\mathcal{N}$-position.

But using only the outcomes does not tell us the outcome of the sum of two $\mathcal{N}$-positions. For example, the sum shown on Figure 8 is an $\mathcal{N}$-position, while the sum shown Figure 9 is a $\mathcal{P}$-position, although both of these are sums of two $\mathcal{N}$-positions!

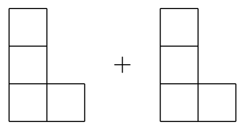

**Fig. 9.** A disjunctive sum of two $\mathcal{N}$-positions of CRAM, which is a $\mathcal{P}$-position.

Table 1 presents the sum of games of known outcomes. But we do not know how to determine the outcome of the sum of two $\mathcal{N}$-positions. We need another tool to help us.

| $+$ | $\mathcal{P}$ | $\mathcal{N}$ |
|---|---|---|
| $\mathcal{P}$ | $\mathcal{P}$ | $\mathcal{N}$ |
| $\mathcal{N}$ | $\mathcal{N}$ | $\mathcal{N}$ or $\mathcal{P}$ |

**Table 1.** Summing impartial games: the outcomes.

We define the equivalence between two games:

**Definition 10** *[Albert et al., 2007] Two games $G$ and $H$ are* equivalent *(denoted by $G \equiv H$) if and only if $G + H$ is a $\mathcal{P}$-position.*

Using Proposition 9, we see that all the games which are $\mathcal{P}$-positions are equivalent. Now, the games which are $\mathcal{N}$-positions are equivalent if and only if their sum is a $\mathcal{P}$-position. We have the following proposition:

**Proposition 11** *For any game $G$, the game $G + G$ is a $\mathcal{P}$-position.*

Indeed, in $G + G$, for any option $o(G)$ of $G$, the first player will play to $G + o(G)$. The second player will only have to play to $o(G) + o(G)$. Always being able to replicate the first player's move, he will play the last move, and win.

**Proposition 12** *$\equiv$ is an equivalence relation.*

Indeed, $\equiv$ is reflexive (we have $G \equiv G$ by Proposition 11), symmetric (if $G \equiv H$, then $G + H$ is a $\mathcal{P}$-position, as is $H + G$, and this implies that $H \equiv G$) and transitive (if $G \equiv H$ and $H \equiv I$, then $G + H$ and $H + I$ are $\mathcal{P}$-positions, thus $G + H + H + I$ is a $\mathcal{P}$-position, and since $H + H$ is a $\mathcal{P}$-position by Proposition 11, it implies that $G + I$ is a $\mathcal{P}$-position, which means that $G \equiv I$).

In order to refine the reasoning, and to be able to determine if the sum of two $\mathcal{N}$-positions is an $\mathcal{N}$-position or a $\mathcal{P}$-position, we attribute numerical values to games, called Grundy values, which will correspond to the equivalence classes of $\equiv$.

The Grundy value of a game is linked to the Grundy values of its options. As the sum of a game $G$ and one of its options $o(G)$ is necessarily an $\mathcal{N}$-position (a winning move being to play from $G+o(G)$ to $o(G)+o(G)$, which is a $\mathcal{P}$-position by Proposition 11), a game can not have the same value than one of its options. Thus, we attribute to a game the smallest possible value which is not shared with any of its options:

**Definition 13** *Let $X$ be a set of nonnegative integers. We define the minimal excluded value, or mex, of $X$ by $mex(X) = min\{a \in \mathbb{N} | \forall x \in X, a \neq x\}$.*

**Definition 14** *[Albert et al., 2007] The* Grundy value *of a game $G$, denoted $\mathcal{G}(G)$, is defined by: $\mathcal{G}(G) = mex(\{\mathcal{G}(o(G)) | o(G)$ is an option of $G\})$.*

We have the following proposition:

**Proposition 15** *[Grundy, 1939] A game $G$ is a $\mathcal{P}$-position if and only if $\mathcal{G}(G) = 0$.*

Now, we can define the equivalence of two games in relation to their values:

**Proposition 16** *[Albert et al., 2007] $G \equiv H$ if and only if $\mathcal{G}(G) = \mathcal{G}(H)$*

Definition 14 allows us to recursively compute the Grundy value of any game.

We now have a way to compute the outcome of a game when it is the sum of two games. However, a game may be decomposed into the sum of three or more games. Thus, we need to define an explicit mean of computing the Grundy value of a sum of games. It has been proven that:

**Proposition 17** *[Albert et al., 2007] $\mathcal{G}(G+H) = \mathcal{G}(G) \oplus \mathcal{G}(H)$, where $\oplus$ is the bitwise exclusive or, XOR, and where $\mathcal{G}(G)$ and $\mathcal{G}(H)$ are written in base 2.*

For example, the three games shown on Figure 10 have the values 0, 1 and 2: the first game is a $\mathcal{P}$-position, and as such has a Grundy value of 0; the second game has only one option, which is the empty grid, and as such has a Grundy value of $mex(\{0\}) = 1$; the third game has two options, which are the second game and the game with only the leftmost and the rightmost cases remaining, and as such has a Grundy value of $mex(\{0; 1\}) = 2$. By Proposition 17, we have that the Grundy value of this sum of games is $0 \oplus 1 \oplus 2 = 3$.

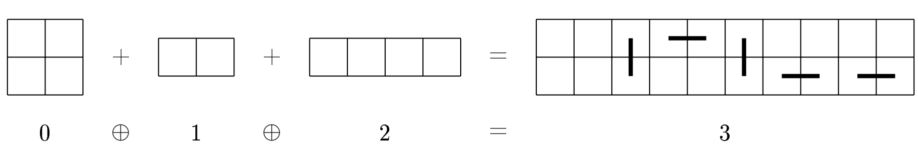


**Fig. 10.** The sum of three games with Grundy values of 0, 1 and 2 is a game with Grundy value 3.

This provides us with a tool to study some impartial games, i.e. those which lend themselves to a decomposition in the form of the sum of smaller games (such as CRAM or NIM). Whether it is the outcome or the Grundy value (which is a stronger result), a position of any impartial game can be thoroughly analyzed, even though the computing cost may be too high. This is why some games are studied under a mathematical perspective: they can be solved in a general case.

The study of impartial games is focused on two main aspects: determining, for any game, its outcome or its Grundy value, and computing the corresponding winning strategy. Studying the time complexity of these aspects is also an important part of the research in that field.

## 2 Octal Games

Octal games are a specific class of combinatorial games, played on heaps of counters.

### 2.1 Definition and Examples

**Definition 18** *[Berlekamp et al., 2001] In the* $0.u_1u_2...u_n...$ *(with* $\forall i, u_i \leq 7$*) game, a player can remove* $i$ *counters from one heap if and only if* $u_i \neq 0$*. Moreover, if* $u_i = \sum_{j=0}^{2} b^i_j 2^j$*, the heap where the counters are removed can be divided into* $j$ *heaps if and only if* $b^i_j \neq 0$*.*

We will give five examples of basic octal games to illustrate the definition.

In the 0.1 game, a player can only empty a heap consisting of one counter, every other move being forbidden. On the left of Figure 11 is an example of a game of 0.1. The non-empty heaps with a size $\geq 2$ are the final positions.

In the 0.2 game, a player can take one counter from a heap if its size is of size at least 2. On the right of Figure 11 is an example of a game of 0.2.

**Fig. 11.** A game of 0.1 (on the left) and a game of 0.2 (on the right).

In the 0.3 game, a player can take one counter from a heap, whether he empties it or not. On the left of Figure 12 is an example of a game of 0.3.

In the 0.4 game, a player can take one counter from a heap, if and only if taking this vertex splits the heap in two smaller, non-empty heaps. On the right of Figure 12 is an example of a game of 0.4.

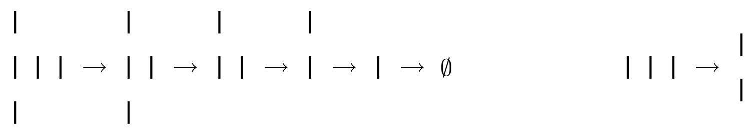

**Fig. 12.** A game of 0.3 (on the left) and a game of 0.4 (on the right).

In the 0.7 game, a player can take one counter from a heap, whether he empties it or not, and can also split the heap in two smaller, non-empty heaps. Figure 13 shows an example of a game of 0.7.

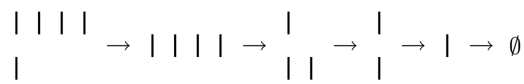

**Fig. 13.** A game of 0.7.

As we can see, an infinite number of octal games can be constructed. More games of the same kind can be constructed, if we allow $u_i \in \mathbb{N}$, those games being called hexadecimal games. However, we shall not expand on them. Besides, octal games can have an infinite number of $u_i$, such as NIM, which is noted 0.333..., since a player can take as many counters as he wants in a heap, provided he does not break that heap in two smaller heaps.

### 2.2 Results

**Definition 19** *[Berlekamp et al., 2001] The* Nim-sequence *of an octal game is the sequence of its Grundy values on a heap of size 0, 1, 2...*

The Nim-sequence is the tool used to analyze octal games. Since the resolution of NIM, the Nim-sequences of many octal games have been studied and computed.

Although octal games seem to be easy, they actually prove to be quite difficult to compute their Nim-sequence in a general case. Even games with very easy rules can be hard to compute.

For example, in the game 0.07, also called Dawson's Kayles, a player can take two counters from a heap, whether he empties it or not, and may split the remaining heap in two smaller, non-empty heaps. We can notice that 0.07 is the same game than CRAM played on a line. The Nim-sequence of 0.07 has a period of 34, after a pre-period of length 68. The 34 first values of the Nim-sequence of 0.07 are: 0112031103322405223301130211045274.

All the games for which the Nim-sequences have been computed exhibit a similar behaviour, which is called *eventual periodicity*: a periodic pattern is observed after a pre-period. For example, the period of the Nim-sequence of the game 0.16 is 149459, with a pre-period of length 105351[Flammenkamp, 2012].

Some other games have not been solved at all, despite having very simple rules. In the 0.007 game, a player can take three counters from a heap, whether it empties the heap or not, and he may split the heap in two smaller, non-empty heaps. The first $2^{28}$ values of the Nim-sequence of this game have been computed, but no periodicity was observed, although only 37 $\mathcal{P}$-positions (Grundy values of 0) have been found.

Even though these results may seem discouraging, some statistical arguments have been formuled, which seem to imply the following conjecture, attributed to Guy:

**Conjecture 20** *[Berlekamp et al., 2001] All finite octal games are eventually periodic.*

### 2.3 Octal Games on Graphs

As seen above, octal games on heaps have been well studied. However, we wanted to extend the study on more complex structures. As heaps are similar to chains in graph theory, it seemed natural to study octal games on graphs. This is a new aspect of the research in this field.

We begin by naturally extending the definition of octal games (Definition 18) to graphs:

**Definition 21** *In the* $0.u_1u_2...u_n$ *(with* $\forall i, u_i \leq 7$*) game on the graph* $G$*, a player can remove* $i$ *vertices from one connected component of* $G$ *if and only if* $u_i \neq 0$*. Moreover, if* $u_i = \sum_{j=0}^{2} b_j^i 2^j$*, the component where the vertices are removed can be divided into* $j$ *components if and only if* $b_j^i \neq 0$*.*

But this extension implies an important question: what do we mean by "removing $i$ vertices"? In the octal games on heaps, there is no notion of connectivity. But, in graphs, there has to be a rule specifying how one is allowed to choose the vertices he will remove from the graph.

At first, we could consider that a player can take the vertices however he wants in the connected component, provided he respects the rules established by the game. But this proves to be rather uninteresting, as it could trivialize games. For example, in the 0.33 game, a player will always be able to take two vertices without disconnecting the component (as long as the connected component contains at least two vertices), making the game equivalent to the 0.33 game on a heap. Thus, we propose a refinement of Definition 21:

**Definition 22** *In the* $0.u_1u_2...u_n$ *(with* $\forall i, u_i \leq 7$*) game on the graph* $G$*, a player can remove* $i$ *vertices from one connected component of* $G$ *if and only if* $u_i \neq 0$ *and the subgraph induced by the vertices is connected. Moreover, if* $u_i = \sum_{j=0}^{2} b_j^i 2^j$*, the component where the vertices are removed can be divided into* $j$ *components if and only if* $b_j^i \neq 0$*.*

The concept of sum naturally arises when playing on different connected components of a graph.

When $G$ is a chain, playing an octal game on $G$ is equivalent to playing an octal game on a heap. Many octal games with $u_i \leq 3$ have been extensively studied and solved on heaps, and as such on chains, hence we began to study 0.03 and 0.33 on different families of graphs.

In the following, we will use the following notations:

**Definition 23** *Given a graph $G$:*

- *$V(G)$ is the set containing the vertices of $G$;*
- *$E(G)$ is the set containing the edges of $G$.*

## 3 0.03 on graphs

### 3.1 Definition

According to Definition 22, when playing 0.03 on a graph $G$, a player can take two adjacent vertices, i.e. an edge, provided that he does not disconnect the graph. We study the outcome of the game in various classes of graphs under the *normal* convention.

### 3.2 Chains

**Definition 24** *A chain $P_m$ (with $m \geq 2$) is a connected graph with $m$ vertices with all but two vertices of degree 2, the last two being of degree 1.*

Playing an octal game on a chain is identical to playing an octal game on one heap of counters, as in Definition 18. Thus, we have the following result:

**Theorem 25** *A chain $P_m$ is an $\mathcal{N}$-position for the 0.03 game if and only if $m \equiv 2 \bmod 4$ or $m \equiv 3 \bmod 4$.*

*Proof.* The proof consists of showing the periodicity of the Nim-sequence for the 0.03 game. It is in the appendix. □

### 3.3 Forests

**Definition 26** *Given a tree $T$, and $e = uv$ one of its edges, we define $T - e$ as the tree with $V(T - e) = V(T) \setminus \{u; v\}$ and $E(T - e) = E(T) \setminus \{e\}$.*

We have the following result:

**Theorem 27** *Given a tree $T$, for the 0.03 game, there are three cases:*

1. *One can not take an edge from $T$ and $T$ is a $\mathcal{P}$-position;*
2. *All options of $T$ are $\mathcal{N}$-positions and $T$ is a $\mathcal{P}$-position;*
3. *All options of $T$ are $\mathcal{P}$-positions and $T$ is an $\mathcal{N}$-position.*

*Proof.* We use induction on $|V(T)|$ to prove the result.

The base cases are the trees such that no move is available: as such, they are $\mathcal{P}$-positions.

Now, suppose that the property holds for a tree $T$. Let us show that the outcome changes when we take any edge. There are two cases:

1. If there is an option $T_1 = T - e$ which is an $\mathcal{N}$-position, then every edge $e'$ one could take from $T$ is still available on $T_1$. Indeed, the only case where taking $e$ would prevent from taking such an $e'$ is if $e$ and $e'$ share a common vertex. However, this only happens if $T = P_3$ and $T_1$ is a single vertex, which is a $\mathcal{P}$-position, and that is not included in this case. We have $T - e' = (T_1 - e') + e$. By induction hypothesis, $T_1 - e'$ is a $\mathcal{P}$-position, and again by induction hypothesis, $T - e' = (T_1 - e') + e$ is an $\mathcal{N}$-position. Since the above equality holds for every $e'$, $T$ is a $\mathcal{P}$-position.
2. Otherwise, for any edge $e$, $T - e$ is a $\mathcal{P}$-position, which implies that $T$ is an $\mathcal{N}$-position.

□

Theorem 27 implies that the strategy does not matter when playing 0.03 on a tree. Indeed, every possible move will be played (or its symmetric will be played, when the tree is reduced to a chain), so the order in which they are played does not matter.

Note that it does not necessarily imply that $T$ will be emptied at the end of a game. Figure 14 shows a non-empty tree from which no move is available.

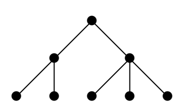

**Fig. 14.** A tree from which no move is available for the 0.03 game.

**Corollary 28** *Given a tree $T$, for the 0.03 game:*

- *If one can not take an edge from $T$, then $\mathcal{G}(T) = 0$;*
- *Otherwise, $\mathcal{G}(T) = 1 - \mathcal{G}(T - e)$ for any option $T - e$ of $T$.*

*Proof.* The only Grundy value for an $\mathcal{N}$-position tree is 1, since, by Theorem 27, one can only move from it to a $\mathcal{P}$-position. □

Playing the game on a forest is exactly playing it on a disjoint union of trees. Thus, by Proposition 17, we have the following result:

**Corollary 29** *Given a forest $F = \bigcup_{i=1}^{n} T_i$, we have $\mathcal{G}(F) = \bigoplus_{i=1}^{n} \mathcal{G}(T_i)$ for the 0.03 game.*

## 3.4 Grids

**Definition 30** *An $m \times n$ grid is a graph $G$ such that:*

- $V(G) = \{u_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n\}$
- $E(G) = \{(u_{i,j}, u_{i,j+1}) | j < n\} \cup \{(u_{i,j}, u_{i+1,j}) | i < m\}$

Thus, an $m \times n$ grid is essentially equivalent to a chessboard of size $m \times n$. Note that we will always consider $n \geq 2$, since an $m \times 1$ grid is the chain $P_m$, which has been solved in a previous section.

We point out that playing 0.03 on a grid is very similar to playing CRAM on a chessboard, with the added connected condition. We will nevertheless focus the rest of our study with a graph theoretical view.

We will need the following definition:

**Definition 31** *Given a connected induced subgraph $G$ of a $m \times n$ grid, we say that a column $j$ (with $j \in [\![1; n]\!]$) of $G$ has size $k$ (with $k \in [\![1; m]\!]$) if there are exactly $k$ vertices in the $j$th column of $G$.*

### 3.4.1 $2 \times n$ Grids

**Definition 32** *An even $(1, 2)$-grid graph is a connected induced subgraph of a $2 \times n$ grid where each block of consecutive columns of size 1 has an even size.*

Figure 15 shows an example of an even $(1, 2)$-grid graph.

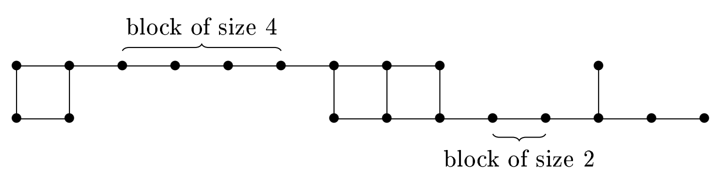


**Fig. 15.** An example of $(1, 2)$-grid graph.

**Lemma 33** *From a non-empty even $(1, 2)$-grid graph, one can only play to an even $(1, 2)$-grid graph.*

*Proof.* The proof is a disjunction of cases. It can be found in the appendix. □

**Theorem 34** *Let $G$ be an even $(1, 2)$-grid graph. At the end of a 0.03 game, $G$ will be empty.*

*Proof.* Let $G$ be an even $(1, 2)$-grid graph. By Lemma 33, we know that as long as $G$ is not empty, both players will always play to an even $(1, 2)$-grid graph. Since each player takes two vertices when playing, it means that there will always be a move available until $G$ is empty. □

**Corollary 35** *Let $G$ be a $2 \times n$ grid. $G$ is an $\mathcal{N}$-position for the 0.03 game if and only if $n$ is odd.*

*Proof.* Let $G$ be a $2 \times n$ grid. By definition, $G$ is an even $(1, 2)$-grid graph, so by Theorem 34, at the end of the game, $G$ will be empty. Since each player takes two vertices when playing, it means that $n$ moves will be played. Thus, the first player will play the last move if and only if $n$ is odd. □

#### 3.4.2 $3 \times n$ Grids

In order to help us conjecture about the outcome of the game on $3 \times n$ grids, we developed a program. Using the algorithm shown in Algorithm 2, we were able to know if a grid was a $\mathcal{P}$-position or an $\mathcal{N}$-position.

**Algorithm 2:** Solve

```
Data: G a position, T a table mapping positions with their outcome
Result: The outcome of the 3 × n grid
if no move is available then
    add the position G with 𝒫 in T
    return 𝒫
if the position is in T then
    return the corresponding outcome
for every move available do
    G′ ← G with the move played
    if Solve(G′, T)) = 𝒫 then
        add the position G with 𝒩 in T
        return 𝒩
add the position G with 𝒫 in T
return 𝒫
```

This algorithm was implemented in C++. We made an extensive use of the standard library, by using the `map` data structure to improve the efficiency of the algorithm: each position met was parsed into a string of 0s and 1s, and was mapped in the structure with its outcome.

Noticing the pattern $(\mathcal{N}\mathcal{N}\mathcal{P}\mathcal{P})*$ in the sequence of outcomes, we saw a relation to the number of possible moves in the grid: the grid is an $\mathcal{N}$ position if and only if $\lfloor \frac{3n}{2} \rfloor$ is odd. Thus, we modified the algorithm to see if the winning player had a strategy consisting of emptying the grid. Now, each position is mapped to its outcome and the maximal number of moves played in a winning strategy trying to empty the grid. We considered that as the winning player tries to empty the grid, the other player tries to prevent it from happening by minimizing the number of moves.

Table 2 shows the result of the execution of the program. We can notice that at the end of a game, the grid is either empty or reduced to a single vertex. This seems to suggest that, as we conjectured and as with the $2 \times n$ grid, a strategy exists to ensure that the grid will be emptied. As shown on Figure 16, there are positions where no move is available and the grid is not emptied, but a winning strategy can always avoid them.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| outcome | $\mathcal{N}$ | $\mathcal{N}$ | $\mathcal{P}$ | $\mathcal{P}$ | $\mathcal{N}$ | $\mathcal{N}$ | $\mathcal{P}$ |
| number of vertices | 3 | 6 | 9 | 12 | 15 | 18 | 21 |
| number of moves played | 1 | 3 | 4 | 6 | 7 | 9 | 10 |

**Table 2.** Number of moves played on a $3 \times n$ grid when applying a winning strategy trying to empty the graph for the 0.03 game.

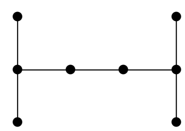

**Fig. 16.** A position where no move is available, but the grid is not emptied.

The program helped us to conjecture the result, but we still need to formally prove it.

**Definition 36** *A* $(1,3)$*-grid graph is a connected induced subgraph of a grid where every column is of size 1 or 3. Moreover, if a column is of size 1, then the vertex in it is not in the middle row.*

Figure 17 shows an example of a $(1,3)$-grid graph.

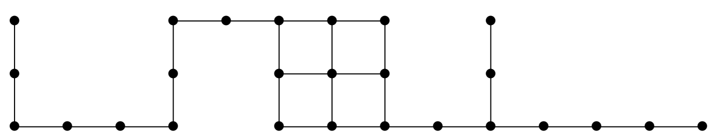

**Fig. 17.** An example of $(1,3)$-grid graph.

**Observation 37** *If a subgraph of a* $3 \times n$ *grid is a chain, then it can be seen a* $(1,3)$*-grid graph. As shown in Figure 18, we can "flatten" it, so that every column is of size 1.*

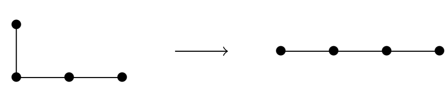

**Fig. 18.** A chain is a $(1,3)$-grid graph.

**Definition 38** *Let* $G$ *be a* $(1,3)$*-grid graph with* $n$ *columns. We define the word* $s(G) = g_1...g_n$ *where* $g_i \in \{1^+; 1^-; 3\}$ *is the size of the ith column. Moreover, if* $g_i = 1^+$ *(resp.* $g_i = 1^-$*), it means that the vertex of the ith column is on the top (resp. bottom) row.*

**Lemma 39** *From a* $(1,3)$*-grid graph* $G$ *with* $|V(G)| \geq 2$*, one can always play to a* $(1,3)$*-grid graph.*

*Proof.* Let $G$ be a $(1,3)$-grid graph with $|V(G)| \geq 2$ and $s(G) = g_1...g_n$. There are two cases:

1. A vertical move is available in a column. In this case, playing it will make the resulting graph a $(1,3)$-grid graph.
2. No vertical move is available. This means that, for every $g_i = 3$, we have $g_{i-1} = 1^+$ and $g_{i+1} = 1^-$, or $g_{i-1} = 1^-$ and $g_{i+1} = 1^+$. This implies that $G$ is a chain, and thus, by Observation 37, a $(1,3)$-grid graph. In this case, playing an horizontal move on either end of the chain will make the resulting graph a $(1,3)$-grid graph.

□

**Lemma 40** *Let* $G$ *be a* $(1,3)$*-grid graph with* $|V(G)| \geq 4$*. For every first move, there exists an answering move resulting in a* $(1,3)$*-grid graph.*

*Proof.* Let $G$ be a $(1,3)$-grid graph $G$ with $|V(G)| \geq 4$ and $s(G) = g_1...g_n$.

First, suppose that the first player played a move on $G$ resulting in a $(1,3)$-grid graph. In this case, the other player can always answer to a resulting $(1,3)$-grid graph, by Lemma 39.

Now, suppose that the first player played a move not resulting in a $(1,3)$-grid graph. According to the proof of Lemma 39, it means that the player played an horizontal move. There are three kind of horizontal moves available in a $(1,3)$-grid graph:

1. Playing in the middle row. In this case, the other player will play in the top row if possible, or in the bottom row if not. At least one of these two moves will be available, and will result in a $(1,3)$-grid graph. Indeed, an horizontal move is available in the middle row if $s(G)$ contains 333, 3331 (whether it is $3331^+$ or $3331^-$ does not matter) or 3333, as shown in Figure 19.

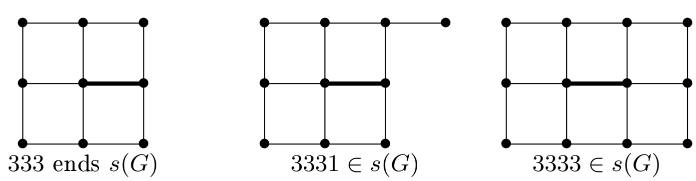


**Fig. 19.** The local topologies from which one can play in the middle row (minus the symmetries).

2. Playing in the top row. In this case, the other player will play in the middle row, just below the first move. This move will always be available, and will result in a $(1,3)$-grid graph. Indeed, an horizontal move is available in the top row if $s(G)$ contains 33, $331^-$, $1^-331^-$, 333, $3331^-$ or 3333, as shown in Figure 20.

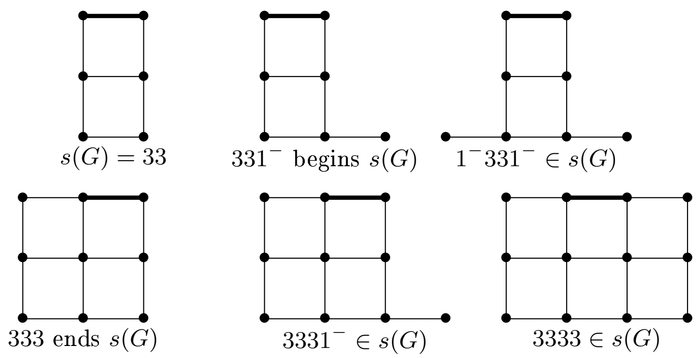


**Fig. 20.** The local topologies from which one can play on the top row (minus the symmetries).

3. Playing in the bottom row. This case is symmetrical to the previous one.

□

**Theorem 41** *A* $(1,3)$*-grid graph* $G$ *is a* $\mathcal{P}$*-position for the 0.03 game if and only if* $\lfloor \frac{|V(G)|}{2} \rfloor$ *is even.*

*Proof.* We use induction on $|V(G)|$ to prove the result. The base cases are $|V(G)| = 0$ and $|V(G)| = 1$. In this case, $\lfloor \frac{|V(G)|}{2} \rfloor = 0$, the first player can not play, thus the graph is a $\mathcal{P}$-position.

Now, suppose we have a $(1,3)$-grid graph $G$ with $|V(G)| \geq 5$. There are two cases:

1. $\lfloor \frac{|V(G)|}{2} \rfloor$ is odd. In this case, the first player can move to a $(1,3)$-grid graph by Lemma 39. By induction hypothesis, the resulting graph is a $\mathcal{P}$-position, thus $G$ is an $\mathcal{N}$-position.
2. $\lfloor \frac{|V(G)|}{2} \rfloor \geq 2$ is even. In this case, after any move of the first player, the second player will be able to play to a $(1,3)$-grid graph $G'$ by Lemma 40. $G'$ satisfies $\lfloor \frac{|V(G')|}{2} \rfloor = \lfloor \frac{|V(G)|-4}{2} \rfloor = \lfloor \frac{|V(G)|}{2} \rfloor - 2$, which is even. By induction hypothesis, $G'$ is a $\mathcal{P}$-position, hence $G$ is a $\mathcal{P}$-position.

□

**Corollary 42** *Let* $G$ *be a* $3 \times n$ *grid. If* $n \equiv 1 \pmod 4$ *or* $n \equiv 2 \pmod 4$*, then* $G$ *is an* $\mathcal{N}$*-position for the 0.03 game. Otherwise,* $G$ *is a* $\mathcal{P}$*-position for the 0.03 game.*

*Proof.* A $3 \times n$ grid is a $(1,3)$-grid graph with $|V(G)| = 3n$. The result holds by applying the Theorem 41, since $\lfloor \frac{3n}{2} \rfloor$ is odd if and only if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$. □

#### 3.4.3 Perspectives

We extended Algorithm 2 to apply it to grids of any size. Since it helped us to conjecture a result for the $3 \times n$ grid, maybe it would also help us for other grids. Table 3 shows the result of the execution of the algorithm on $4 \times n$.

| $n$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| number of moves played on the $4 \times n$ grid | 4 | 6 | 8 | 10 |

**Table 3.** Number of moves played on $4 \times n$ grids.

We see that, again, the grid is emptied at the end of the game. We state the Conjecture 43.

**Conjecture 43** *In an $m \times n$ grid, there exists a winning strategy such that $\lfloor \frac{mn}{2} \rfloor$ moves will be played.*

In other words, we are inclined to think that, for the 0.03 game, there is a winning strategy where every possible move will be played.

Trying to prove or infirm it, or at least to study it for specific values of $m$, is an interesting perspective. The idea could be to try finding other structures, similar to the $(1, 3)$-grid graph, which could guarantee that no blocking situation would happen.

## 4 0.33 on graphs

### 4.1 Definition

According to Definition 22, when playing 0.33 on a graph $G$, a player can take either one vertex or two adjacent vertices provided that he does not disconnect the graph. We study the outcome of the game in some classes of trees under the *normal* convention.

### 4.2 Chains

As previously said, playing an octal game on a chain is identical to playing an octal game on one heap of counters. Thus, we have the following result:

**Theorem 44** *A chain $P_m$ is a $\mathcal{P}$-position for the 0.33 game if and only if $m \equiv 0 \bmod 3$. Moreover, we have $\mathcal{G}(P_m) = m \bmod 3$.*

*Proof.* The proof consists of showing the periodicity of the Nim-sequence for the 0.33 game. It is in the appendix. □

### 4.3 $k$-podes

**Definition 45** *A $k$-pode $Pod(l_1, ..., l_k)$ is a graph with a central vertex, on which are appended $k$ chains of length $l_1, ..., l_k$.*

Figure 21 shows an example of $k$-pode. The $k$-pode being the simplest tree after the chain, we hope to find interesting results.

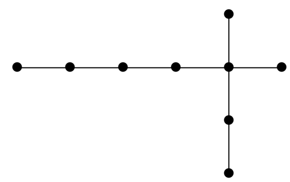

**Fig. 21.** The 4-pode $Pod(1, 1, 2, 4)$.

**Observation 46** *The 0-pode $Pod(0)$ is a single vertex.*
*The 1-pode $Pod(l)$ is the chain $P_{l+1}$.*
*The 2-pode $Pod(l_1, l_2)$ is the chain $P_{l_1+l_2+1}$.*

**Lemma 47** *Let $P$ be a $k$-pode, with $P \neq Pod(0)$ and $P \neq Pod(1)$. If $P$ is an $\mathcal{N}$-position for the 0.33 game, then there is a winning move which does not involve taking the central vertex.*

*Proof.* Let $P$ be a $k$-pode ($P \neq Pod(0)$ and $P \neq Pod(1)$).

First, suppose that $P$ is not a chain, i.e. $P = Pod(l_1, l_2, l_3)$, with $l_1, l_2, l_3 > 0$. Then, no winning move may involve taking the central vertex, as such a move would be illegal.

Now, let $P$ be a chain of length $n$. Since $P$ is an $\mathcal{N}$-position and $P \notin \{Pod(0); Pod(1)\}$, we can assume that $n \geq 4$. If the winning move involves taking the central vertex, then the winning move can be taking one or two vertices including the central vertex. Hence the central vertex is the first or the second vertex of the chain and, since $n \geq 4$, the winning move can be replicated on the other end of the chain without taking the central vertex. □

We now prove that one can add a $P_3$ to a leaf or to the central vertex of a $k$-pode without changing the issue and the grundy number of the game. We first prove a technical lemma.

**Lemma 48** $\mathcal{G}(Pod(0)) = \mathcal{G}(Pod(1, 1, 1))$

*Proof.* We prove that $Pod(0) + Pod(1, 1, 1)$ is a $\mathcal{P}$-position. Let us show that the second player always has an answer to the first player's possible moves:

- If the first player empties $Pod(0)$, then the second player's answer is to take one vertex from $Pod(1, 1, 1)$, leaving $P_3$ which is a $\mathcal{P}$-position.
- If the first player takes one vertex from $Pod(1, 1, 1)$, then the second player's answer is to empty $Pod(0)$, leaving $P_3$ which is a $\mathcal{P}$-position.

Thus, no matter what move the first player plays, he plays in an $\mathcal{N}$-position. This implies that $Pod(0) + Pod(1,1,1)$ is a $\mathcal{P}$-position. □

The proofs for the other technical lemmas will be similar, so they will be placed in the appendix.

**Lemma 49** *$\mathcal{G}(Pod(1,1,3)) = 0$, i.e. $Pod(1,1,3)$ is a $\mathcal{P}$-position.*

**Lemma 50** *For any $i \in [\![1;k]\!]$, $\mathcal{G}(Pod(l_1, ..., l_i, ..., l_k)) = \mathcal{G}(Pod(l_1, ..., l_i{+}3, ...l_k))$.*

Note that we allow $l_i = 0$.

*Proof.* Let $P = Pod(l_1, ..., l_i, ..., l_k)$ and $P' = Pod(l_1, ..., l_i + 3, ...l_k)$. We show that $P + P'$ is a $\mathcal{P}$-position. We reason by induction on $|V(P)|$.

The base cases are the following:

- If $P$ is empty (resp. $P = P_1$, resp. $P = P_2$), then $P' = P_3$ (resp. $P' = P_4$, resp. $P' = P_5$), so $\mathcal{G}(P) = \mathcal{G}(P')$ by Theorem 44.
- If $P = P_3$, then either $P' = P_5$ and we have the result by Theorem 44, or $P' = Pod(1,1,3)$ and we have the result by Lemma 49.

From a certain position $P + P'$, the second player always has an answer to the first player's move:

- If the first player takes one (resp. two) vertex from the new chain in $P'$, then the second player takes two (resp. one) vertices from it, leaving $P + P$ which is a $\mathcal{P}$-position by Proposition 11.
- If the first player plays elsewhere on $P'$, the second player answers by playing the same move on $P$. This will always be possible. By induction hypothesis, the new position will be a $\mathcal{P}$-position.
- If the first player plays on $P$, there are two cases:
  1. The first player does not take the central vertex. In this case, the second player can replicate the move on $P'$, allowing us to invoke the induction hypothesis.
  2. The first player takes the central vertex. This implies that $P = P_m$ with $m \geq 4$. Then, as said in the proof of Lemma 47, the second player will always be able to replicate the first player's move on $P'$, by playing the symmetrical move. By induction hypothesis, the new position will be a $\mathcal{P}$-position.

□

This allows us to deduce this result:

**Corollary 51** $\mathcal{G}(Pod(l_1, ..., l_k)) = \mathcal{G}(Pod(l_1 \bmod 3, ..., l_k \bmod 3))$

From Corollary 51, we know that all chains of length $3p$ can be reduced to 0, all chains of length $3p+1$ can be reduced to 1, and all chains of length $3p+2$ can be reduced to 2. Thus, it suffices to study the Grundy values of the game on $Pod(l_1, ..., l_k)$ with $\forall i \in [\![1;k]\!]$, $l_i \in \{1;2\}$.

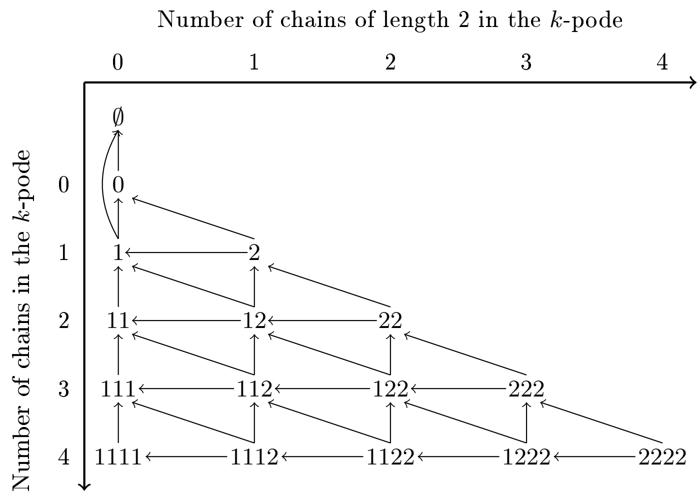


**Fig. 22.** First five rows of the table of positions

We build a table of positions: the rows stand for the number of chains attached to the central vertex, and the columns stand for the number of chains of length 2. Figure 22 shows the first five rows of this table (the top positions are the empty graph and the 0-pode with only the central vertex).

We can notice that, from a given position, a player can always move in three directions: up (reducing a chain of length 1 to 0 by taking a vertex), left (reducing a chain of length 2 to 1 by taking a vertex), and up left (reducing a chain of length 2 to 0 by taking two vertices). The only position from which another move is available is from $Pod(1)$: taking two vertices results in the empty graph.

First, let us study the outcomes of each of these positions. We proceed inductively from the top lines:

- The empty graph is a $\mathcal{P}$-position, since from it one can not play;
- $Pod(0)$ is an $\mathcal{N}$-position, since it represents a single vertex: the player has to take it in order to win;
- $Pod(1)$ is an $\mathcal{N}$-position, since it represents a chain of length 1 attached to the central vertex: the player has to take the two adjacent vertices in order to win;
- $Pod(2)$ is a $\mathcal{P}$-position, since from it a player can only move to winning positions ($Pod(0)$ or $Pod(1)$).

By continuing to apply this reasoning, we can deduce the table of outcomes shown on Figure 23.

We are now able to deduce this result :

**Lemma 52** *For $p \geq 1$, the $(2p)$th row of the table will be of the form: $\mathcal{PNNN}$ $(\mathcal{PN})^*$, and the $(2p+1)$th row of the table will be of the form: $\mathcal{NNP}(\mathcal{N})i*$*

*Proof.* We use induction on $p$ to prove the result. The complete proof can be found in the appendix. □

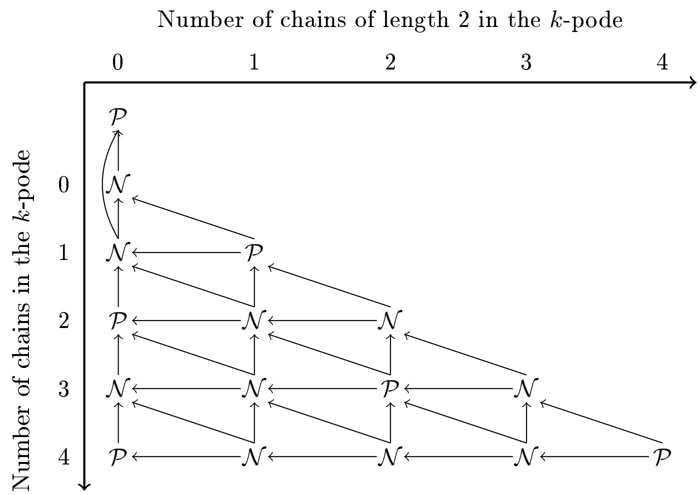


**Fig. 23.** First five rows of the table of positions

From Lemma 52, we are now able to determine precisely the winning positions for the 0.33 game on any $k$-pode:

**Theorem 53** *Let $P$ be a k-pode with $n_1$ (resp. $n_2$) being the number of chains of length $3p+1$ (resp. $3p+2$) in $P$. $P$ is an $\mathcal{N}$-position for the 0.33 game if one of these four conditions holds :*

1. $n_1 = n_2 = 0$
2. $n_2$ *is odd and* $(n_1, n_2) \neq (0, 1)$
3. $n_2 = 2$ *and* $n_1$ *is even*
4. $n_2 \neq 2$ *is even and* $n_1$ *is odd*

A table of the Grundy values for the $k$-podes can be found using the same arguments, as shown in Figure 34, in the appendix.

### 4.4 Bipodes

**Definition 54** *A $(k, l, m)$-bipode $Bipod(l_1, ..., l_k; l_{k+1}, ...l_{k+l}; m)$ is a graph constituted of the k-pode $Pod(l_1, ..., l_k)$ and the l-pode $Pod(l_{k+1}, ...l_{k+l})$ whose central vertices are connected by a chain of $m$ edges.*

In other terms, a bipode is a tree with exactly two vertices of degree 3 or more. Figure 24 shows an example of bipode.

In a bipode, we can reduce the middle chain modulo 3:

**Lemma 55** $\mathcal{G}(Bipod(l_1, ..., l_{k+l}; m)) = \mathcal{G}(Bipod(l_1, ..., l_{k+l}; m \bmod 3))$

*Proof.* We will prove that adding three edges to the chain does not change the Grundy value of the bipode, the result being a generalization of this.

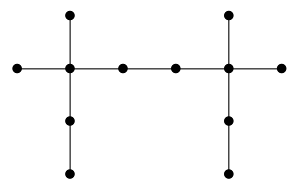

**Fig. 24.** The $(3,3,3)$-bipode $Bipod(1,1,2;1,1,2;3)$.

Let $B$ be a $(k,l,m)$-bipode, with $P$ (resp. $Q$) the $k$-pode (resp. the $l$-pode) at one extremity of the chain of length $m$, as shown in Figure 25. Let $B'$ be the $(k,l,(m+3))$-bipode obtained by adding three edges to the chain connecting $P$ and $Q$ in $B$, as shown in Figure 26.

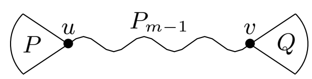


**Fig. 25.** $B$ with $P_{m-1}$ the chain of $m$ edges.

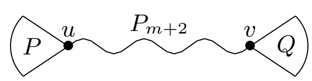


**Fig. 26.** $B$ with $P_{m+2}$ the chain of $m+3$ edges.

We show that $B+B'$ is a $\mathcal{P}$-position. We use induction on the size of $B$.

The base cases are when $P$ is empty or a single vertex. In this case, $B$ and $B'$ are $(l+1)$-podes, and the result is covered by Lemma 50.

Now, we show that, for a certain $B+B'$, the second player always has an answer to the first player's move. We can assume that both $P$ and $Q$ have at least two vertices. Hence, the first player is unable to play on the middle chain. Thus, the first player can play either on $P$ or $Q$, in either of the two bipodes. The second player will replicate the same move on the other bipode. By induction hypothesis, we have the result. □

We will now prove that we can add a chain of length 3 in a bipode to a leaf or to one central vertex without changing the Grundy value of the graph. We first need a few technical lemmas. The proofs are in the appendix, and follow the same reasoning than the proofs for the technical lemmas of the $k$-pode.

**Lemma 56** $\mathcal{G}(Bipod(1,1;1,1;1)) = 0$, *i.e.* $Bipod(1,1;1,1;1)$ *is a* $\mathcal{P}$*-position.*

**Lemma 57** *Let* $B$ *and* $B'$ *be the two graphs shown on Figure 27 (*$P$ *is a* $k$*-pode with* $u$ *as its central vertex). We have* $\mathcal{G}(B) = \mathcal{G}(B')$.

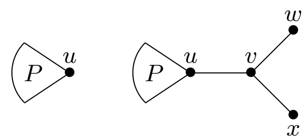


**Fig. 27.** $B$ and $B'$

**Lemma 58** *Let $B$ and $B'$ be the two graphs shown on Figure 28 ($P$ is a $k$-pode with $u$ as its central vertex). We have $\mathcal{G}(B) = \mathcal{G}(B')$.*

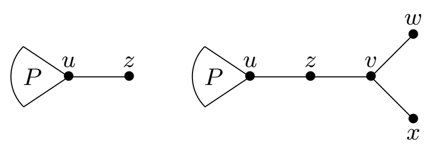


**Fig. 28.** $B$ and $B'$

We are now ready to generalize Lemma 50 to bipodes:

**Lemma 59** $\mathcal{G}(Bipod(l_1, ..., l_i, ..., l_{k+l}; m)) = \mathcal{G}(Bipod(l_1, ..., l_i + 3, ..., l_{k+l}; m))$

As in Lemma 50, we allow $l_i = 0$.
This implies:

**Corollary 60** $\mathcal{G}(Bipod(l_1, ..., l_{k+l}; m)) = \mathcal{G}(Bipod(l_1 \bmod 3, ..., l_{k+l} \bmod 3; m))$

We are now able to compute the Grundy values of a bipode, by using induction.

**Theorem 61** *Let $B_{a_1,a_2,b_1,b_2}$ be a $(k, l, m)$-bipode, with $P$ (resp. $Q$) the $k$-pode (resp. the $l$-pode) at one extremity of the chain of length $m$, with $a_1$ (resp. $a_2$) the number of chains of length 1 (resp. 2) attached to the central vertex of $P$ ($b_1$ and $b_2$ are defined the same way on $Q$).*
$\mathcal{G}(B_{a_1,b_1,a_2,b_2}) = mex \quad ($

- $\mathcal{G}(B_{a_1-1,a_2,b_1,b_2})$ *if* $a_1 > 0$
- $\mathcal{G}(B_{a_1,a_2,b_1-1,b_2})$ *if* $b_1 > 0$
- $\mathcal{G}(B_{a_1,a_2-1,b_1,b_2})$ *if* $a_2 > 0$
- $\mathcal{G}(B_{a_1,a_2,b_1,b_2-1})$ *if* $b_2 > 0$
- $\mathcal{G}(B_{a_1+1,a_2-1,b_1,b_2})$ *if* $a_2 > 0$
- $\mathcal{G}(B_{a_1,a_2,b_1+1,b_2-1})$ *if* $b_2 > 0$
- $\mathcal{G}(Pod(1, ..., 1, 2, ...2, m - 1))$ *(with $b_1$ chains of length 1 and $b_2$ chains of length 2) if* $a_1 = 1$ *and* $a_2 = 0$
- $\mathcal{G}(Pod(1, ..., 1, 2, ...2, m - 1))$ *(with $a_1$ chains of length 1 and $a_2$ chains of length 2) if* $b_1 = 1$ *and* $b_2 = 0$

)

*Proof.* The terminal values are found in the table of $k$-podes and chains. The relation comes from the moves which can be played from $B_{a_1,a_2,b_1,b_2}$. □

Thanks to this formula, we can compute a four-dimensional table of Grundy values for the bipodes. A table for the Grundy values of $(k, l, 1)$-bipodes (resp. $(k, l, 2)$-bipodes) can be found Table 4 (resp. Table 5) in the appendix. It has been computed using a recursive program written in C++. We notice that a periodicity of the values establishes itself.

We also notice that $(k, l, 1)$-bipodes behave almost like a sum of $k$-podes in terms of Grundy values. Some of the results can be induced this way: if one of the two $k$-podes is a $\mathcal{P}$-position, then the bipode will have the Grundy value of the other one: the winning player will apply his strategy on the other $k$-pode, and answer any move of the other player on the $\mathcal{P}$ $k$-pode by the winning strategy in it. If the bipode is large enough, then its Grundy value can be computed as a sum of two $k$-podes. Some terminal cases are problematic, but they can be treated separately.

This allows us to compute the Grundy value of a $(k, l, 1)$-bipode by using the Grundy values of its two $k$-podes.

However, the same can not be said of $(k, l, 2)$-bipodes. Instead, it could be interesting to study similar games, such as 0.32 or 0.23, and try to express the Grundy value of a $(k, l, 2)$-bipode for 0.33 by using the Grundy value of its two $k$-podes for those other games.

### 4.5 Perspectives

The main argument used to solve the 0.33 game for the $k$-podes and bipodes was to reduce all the chains ending in a leaf and the internal chain between two central nodes to their modulo 3, which greatly reduced the search space, and allowed to have algorithms to compute the Grundy value.

However, this reduction is not possible in the trees:

**Observation 62** *One cannot add a $P_3$ to any vertex of a bipode without changing the Grundy value (and even without changing the output). Indeed, the bipode of Figure 29 is an $\mathcal{N}$-position, but appending a $P_3$ to $u$ changes it into a $\mathcal{P}$-position.*

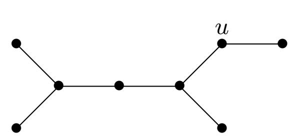


**Fig. 29.** Counter-example for trees.

This means that there is still work to do for the 0.33 game on trees. An idea could be to use related octal games, such as 0.32, 0.23 or 0.22, to help us to compute Grundy values for caterpillars.

An other interesting problem could be to study the 0.33 game on $3k \times n$ grids. We state the following conjecture:

**Conjecture 63** *Any $3k \times n$ grid is a $\mathcal{P}$-position for the 0.33 game.*

This comes from the fact that there are $3kn$ vertices in a $3k \times n$ grid, and the second player may always be able to take one vertex (resp. two vertices) if the first player takes two vertices (resp. one vertex). The idea could be to find a structure ensuring that the second player could always take two vertices after the first player taking one vertex, as what was made for the 0.03 game.

## Conclusion

Combinatorial games are a vast domain of research, with many open problems and possible extensions. During this internship, we extended the concept of octal games to graphs, and studied two of these games on some classes of graphs.

For the 0.03 game, the results found on grids are promising. The implementation of the algorithm allowed us to state a conjecture which could be interesting, and we could try and find a way to extend the game to other classes of graphs. In particular, the case of general grids is the next step that could be solved.

The 0.33 game could be disappointing: the reduction we used to $k$-podes and bipodes can not be generalized to trees, even though interesting results were found for particular classes of trees. However, other classes of graphs may be solved for this game.

An other perspective would be to develop the field of octal games on graphs, by studying other games. It could also be interesting to try and generalize Conjecture 20 on graphs.

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# A Proofs for 0.03

### Proof of Theorem 25

*A chain $P_m$ is an $\mathcal{N}$-position for the 0.03 game if and only if $m \equiv 2 \bmod 4$ or $m \equiv 3 \bmod 4$.*

*Proof.* We reason by induction on $m$. The base case is:

- $m = 0$ (resp. $m = 1$). This means $P_m$ is an empty chain (resp. a single vertex). In this case, the first player can not play, and the chain is a $\mathcal{P}$-position.

Let $m \geq 2$. Suppose that the result holds for all $i < m$. Let us study the outcome of $P_{m+2}$.

From $P_{m+2}$, one can only take two vertices on either side of $P_m$, since one can not disconnect the chain. Thus, the only available move from $P_{m+2}$ is to play to $P_m$.

If $m + 2 \equiv 2 \bmod 4$ (resp. $m + 2 \equiv 3 \bmod 4$), then $m \equiv 0 \bmod 4$ (resp. $m \equiv 1 \bmod 4$), which imply that $P_m$ is a $\mathcal{P}$-position by induction hypothesis. Thus, $P_{m+2}$ is an $\mathcal{N}$-position.

Conversely, if $m+2 \equiv 0 \bmod 4$ (resp. $m+2 \equiv 1 \bmod 4$), $m \equiv 2 \bmod 4$ (resp. $m \equiv 3 \bmod 4$), $P_m$ is an $\mathcal{N}$-position by induction hypothesis, which implies that $P_{m+2}$ is a $\mathcal{P}$-position. □

### Proof of Lemma 33

*From a non-empty even $(1, 2)$-grid graph, one can only play to an even $(1, 2)$-grid graph.*

*Proof.* Let $G$ be an even $(1, 2)$-grid graph. We note that a move is always available if $G$ is not empty: if there is a column of size 2 on either side of $G$, then a vertical move is available ; otherwise, as the number of consecutive columns of size 1 is even, an horizontal move on either side of $G$ will be available.

Now, we consider the two possible kinds of move in $G$:

1. Vertical moves, if available, will only appear on either side of $G$, and will result in an even $(1, 2)$-grid graph.
2. Horizontal moves, since they take two adjacent vertices and can not disconnect $G$, they result in an even $(1, 2)$-grid graph.

□

## B Proofs for 0.33

### Proof of Theorem 44

*A chain $P_m$ is a $\mathcal{P}$-position for the 0.33 game if and only if $m \equiv 0 \bmod 3$. Moreover, we have $\mathcal{G}(P_m) = m \bmod 3$.*

*Proof.* We use induction on $m$ to prove the result. The base cases are:

- $m = 0$. In this case, $P_m$ is the empty graph, and thus is a $\mathcal{P}$-position. This implies that $\mathcal{G}(P_0) = 0$.
- $m = 1$. In this case, $P_m$ is a single vertex, and thus is an $\mathcal{N}$-position: the first player only has to take the vertex to win. From it, one can only play to the empty graph, thus $\mathcal{G}(P_1) = mex(\mathcal{G}(P_0)) = mex(0) = 1$.
- $m = 2$. In this case, the first player has to take the two vertices to win, thus the graph is an $\mathcal{N}$-position. From it, one can play to both the empty graph and the single vertex, thus $\mathcal{G}(P_2) = mex(\mathcal{G}(P_0), \mathcal{G}(P_1)) = mex(0, 1) = 2$.

Now, from a chain $P_m$, one can play to either $P_{m-1}$ or $P_{m-2}$. Thus, $\mathcal{G}(P_m) = mex(\mathcal{G}(P_{m-1}), \mathcal{G}(P_{m-2}))$. By induction hypothesis, this means that $\mathcal{G}(P_m) = mex(m-1 \bmod 3, m-2 \bmod 3) = m \bmod 3$. Since a game is a $\mathcal{P}$-position if and only if its Grundy value is 0, it means that $P_m$ is a $\mathcal{P}$-position if and only if $m \equiv 0 \bmod 3$. □

### Proof of Lemma 49

*$\mathcal{G}(Pod(1, 1, 3)) = 0$, i.e. $Pod(1, 1, 3)$ is a $\mathcal{P}$-position.*

*Proof.* From $Pod(1, 1, 3)$, the first player only has two moves available:

- Taking one vertex from either of the two chains of length 1. In this case, the graph is reduced to $P_5$, which is an $\mathcal{N}$-position by Theorem 44.
- Taking one (resp. two) vertex from the chain of length 3. In this case, the second player takes two (resp. one) vertices from the same chain, leaving $P_3$, which is a $\mathcal{P}$-position.

□

### Proof of Lemma 52

*For $p \geq 1$, the $(2p)$th row of the table will be of the form: $\mathcal{PNNN}$ $(\mathcal{PN})^*$, and the $(2p+1)$th row of the table will be of the form: $\mathcal{NNP}(\mathcal{N})i*$*

*Proof.* We argue by induction on $p = \lfloor k/2 \rfloor$. The base cases are the rows 2 and 3 of the table.

Assume that the result holds for a certain $p$. Figure 30 shows the $(2p)$th and $(2p+1)$th rows.

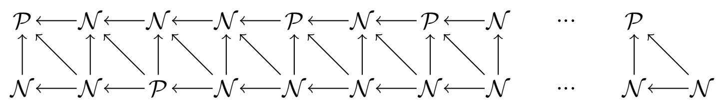

**Fig. 30.** The $(2p)$th and $(2p+1)$th rows. The $(2p)$th row ends with a $\mathcal{P}$ since there is an odd number of positions in it.

Now, we can reason inductively: if, from a position, one can play to a $\mathcal{P}$-position, then this position is an $\mathcal{N}$-position. Conversely, if from a position one can only play to $\mathcal{N}$-positions, then it is a $\mathcal{P}$-position. Figure 31 shows the result of this reasoning, corresponding to the formula.

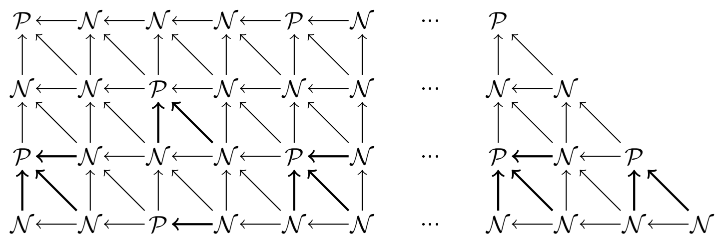

**Fig. 31.** The rows $(2p)$ to $(2p+3)$. In thick are the moves from which one can play from an $\mathcal{N}$-position to a $\mathcal{P}$-position.

□

### Proof of Lemma 56

*$\mathcal{G}(Bipod(1,1;1,1;1)) = 0$, i.e. $Bipod(1,1;1,1;1)$ is a $\mathcal{P}$-position.*

*Proof.* From $Bipod(1,1;1,1;1)$, as shown in Figure 32, one can only play to a graph which can be seen as $Pod(1,1,2)$. We can refer to the Figure 23, and notice that $Pod(1,1,2)$ is an $\mathcal{N}$-position. Thus, $Bipod(1,1;1,1;1)$ is a $\mathcal{P}$-position.

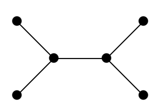

**Fig. 32.** $Bipod(1,1;1,1;1)$

□

### Proof of Lemma 57

*Let $B$ and $B'$ be the two graphs shown on Figure 27 ($P$ is a $k$-pode with $u$ as its central vertex). We have $\mathcal{G}(B) = \mathcal{G}(B')$.*

*Proof.* We show that $B + B'$ is a $\mathcal{P}$-position. We use induction on the size of $B$. The base cases are:

- $B$ is empty, thus $B' = P_3$. $B + B' = P_3$, which is a $\mathcal{P}$-position.
- $B$ is a vertex, thus $B' = Pod(1, 1, 1)$. We know by Lemma 48 that $\mathcal{G}(B) = \mathcal{G}(B') = 1$.
- $B$ is $Pod(1)$, thus $B' = Pod(1, 1, 2)$. We only have to refer to Figure 34 to know that $\mathcal{G}(B) = \mathcal{G}(B') = 2$.
- $B$ is $P_3$ with $u$ as its central vertex, thus $B' = Bipod(1, 1; 1, 1; 1)$. By Lemma 56, $B'$ is a $\mathcal{P}$-position, as is $B$, so $\mathcal{G}(B) = \mathcal{G}(B') = 0$.

From a certain position $B + B'$, the second player always has an answer to the first player's moves:

- If the first player plays on $P$ on $B'$, the second player plays the same move on $B$. By induction hypothesis, the resulting graph will be a $\mathcal{P}$-position.
- If the first player takes $w$ (resp. $x$), the second player takes $v$ and $x$ (resp. $v$ and $w$). The resulting graphs will be $B + B$, which is a $\mathcal{P}$-position.
- If the first player plays on $B$, there are two cases:
  1. The first player does not take $u$. In this case, the second player can replicate the move on $B'$, allowing us to invoke the induction hypothesis.
  2. The first player takes $u$. This implies that $P = P_m$ with $m \geq 4$, or that $P = P_3$ with $u$ as its central vertex. Then, by Lemma 47, the second player will always be able to replicate the first player's move on $B'$, by playing the symmetrical move. By induction hypothesis, the new position is a $\mathcal{P}$-position.

□

**Proof of Lemma 58**

*Let $B$ and $B'$ be the two graphs shown on Figure 28 ($P$ is a $k$-pode with $u$ as its central vertex). We have $\mathcal{G}(B) = \mathcal{G}(B')$.*

*Proof.* We show that $B + B'$ is a $\mathcal{P}$-position. We use induction on the size of $B$. The base cases are:

- $B$ is a single vertex (i.e. $P$ is empty), thus $B' = Pod(1, 1, 1)$. We know by Lemma 48 that $\mathcal{G}(B) = \mathcal{G}(B') = 1$.
- $B$ is two vertices (i.e. $P$ is a single vertex), thus $B' = Pod(1, 1, 2)$. If we refer to Figure 23, we know that $\mathcal{G}(B) = \mathcal{G}(B') = 2$.
- $B$ is $P_3$ (i.e. $P$ is two vertices), thus $B' = Pod(1, 1)$ (by Lemma 50, we know that the chain of length 3 can be deleted without changing the grundy value). If we refer to Figure 23, we know that $B'$ is a $\mathcal{P}$-position, and since $B = P_3$ is also a $\mathcal{P}$-position, we have that $B + B'$ is a $\mathcal{P}$-position.

- $B = Pod(1,1,1)$, thus $B' = Bipod(1,1;1,1;2)$. From $B + B'$, the first player can only take one vertex from either of the two graphs. Whether he takes one vertex from the $k$-pode (reducing it to $P_3$) or from the bipode (reducing it to $Pod(1,1,3)$, which has the same grundy value than $Pod(1,1)$), the second player only has to take one vertex from the other graph. Thus, the game will be reduced to $P_3 + Pod(1,1)$, which, as seen above, is a $\mathcal{P}$-position.

The end of the proof is exactly the same than for Lemma 57, with exactly the same cases: the first player playing on $P$ in $B'$, playing on $w$ or $x$, playing on $P$ in $B$ and not taking $u$, and playing on $P$ in $B$ and taking $u$.

Note that the case where the first player can take both $u$ and $z$ has been covered in the base cases. □

**Proof of Lemma 59**

$$\mathcal{G}(Bipod(l_1, ..., l_i, ..., l_{k+l}; m)) = \mathcal{G}(Bipod(l_1, ..., l_i + 3, ..., l_{k+l}; m))$$

*Proof.* Thanks to Lemma 55, we only have to prove the result on the $k, l, 1$-bipodes and the $k, l, 2$-bipodes (the $k, l, 0$-bipodes being $(k+l)$-podes, their case is already covered by Lemma 50).

Let $B = Bipod(l_1, ..., l_i, ..., l_{k+l}; m)$, with $P$ (resp. $Q$) the $k$-pode (resp. the $l$-pode) at one extremity of the chain of length $m \in \{1; 2\}$, and $B' = Bipod(l_1, ..., l_i + 3, ..., l_{k+l}; m)$. We suppose, without loss of generality, that $i \in [\![k+1; k+l]\!]$. We reason by induction on $|V(B)|$.

First, we consider the base cases:

- If $Q$ is empty (resp. a single vertex), then $B$ is an $(k+1)$-pode, and the result holds by Lemma 50.
- If $Q$ is a chain, and $l_i = 0$, then replicating the first player's move will always be possible, except in the cases shown Figure 33 (above is when $m = 1$ below is when $m = 2$).

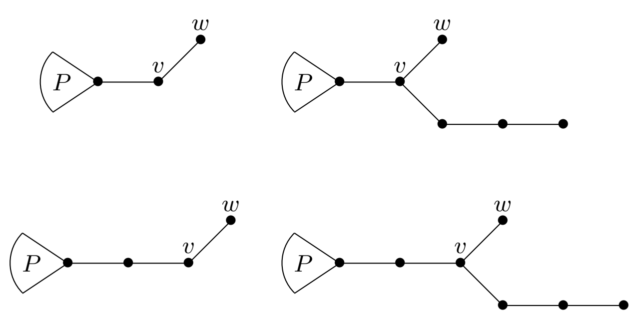


**Fig. 33.** Cases where the second player can not replicate the first player's move.

In those cases, if the first player takes both $v$ and $w$ on $B$, then the second player is unable to replicate the move on $B'$. The strategy is then to take two vertices from the new chain. By Lemma 57 (when $m = 1$) and Lemma 58 (when $m = 2$), we have $\mathcal{G}(B) = \mathcal{G}(B')$.

Now, we consider a certain $B$ and $B'$. We show that $B + B'$ is a $\mathcal{P}$-position. For a certain position $B + B'$, we show that the second player always has an answer to the first player's move:

- If the first player plays on $P$ on either of the two graphs, then the second player answers by replicating his move on the other graph, allowing us to invoke the induction hypothesis.
- If the first player takes one (resp. two) vertex from the new chain on $B'$, then the second player answers by taking two (resp. one) vertices from it, leaving $B + B$ which is a $\mathcal{P}$-position.
- If the first player plays on $Q$ on $B'$, then the second player answers by replicating his move on $B$, which will always be possible, allowing us to invoke the induction hypothesis.
- If the first player plays on $Q$ on $B$, he will not be able to take its central vertex, since $Q$ is not a chain. Thus, his move can be replicated on $B'$, allowing us to invoke the induction hypothesis.

□

## C Grundy values for the $k$-podes

Figure 34 shows the first rows of the Grundy values for the 0.33 game on $k$-podes.

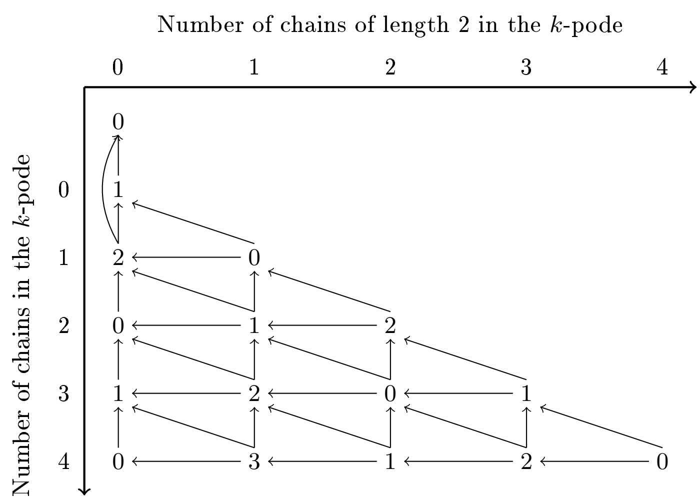


**Fig. 34.** First five rows of the table of Grundy values

We have the following result:

**Theorem 64** *For $p \geq 3$, the $(2p-1)$th row of the table will be of the form: 0312(03)*, and the $(2p)$th row of the table will be of the form: 1203(12)*.*

The proof unfolds exactly the same way than the proof for Lemma 52.

## D Grundy values for the $k, l, 1$-bipodes

Let $B_{a_1,a_2,b_1,b_2}$ be a $(k,l,1)$-bipode, with $P$ (resp. $Q$) the $k$-pode (resp. the $l$-pode) at one extremity of the chain of length 1, with $a_1$ (resp. $a_2$) the number of chains of length 1 (resp. 2) attached to the central vertex of $P$ ($b_1$ and $b_2$ are defined the same way on $Q$).

Table 4 contains the Grundy values of $B_{a_1,a_2,b_1,b_2}$ for $(a_1,a_2,b_1,b_2) \leq (6,6,6,6)$. We stopped the computation at this point since the periodicity establishes itself, due to the way the recursive formula exposed in Theorem 61 works.

## E Grundy values for the $k, l, 2$-bipodes

Let $B_{a_1,a_2,b_1,b_2}$ be a $(k,l,2)$-bipode, with $P$ (resp. $Q$) the $k$-pode (resp. the $l$-pode) at one extremity of the chain of length 2, with $a_1$ (resp. $a_2$) the number of chains of length 1 (resp. 2) attached to the central vertex of $P$ ($b_1$ and $b_2$ are defined the same way on $Q$).

Table 5 contains the Grundy values of $B_{a_1,a_2,b_1,b_2}$ for $(a_1,a_2,b_1,b_2) \leq (6,6,6,6)$. We stopped the computation at this point since the periodicity establishes itself, due to the way the recursive formula exposed in Theorem 61 works.

$b_2 = 0$

| $a_2$ | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 3 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 2 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 3 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 2 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 3 | 1 | 2 | 3 | 2 | 3 | 2 |
| 2 | 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| 3 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 3 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 2 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 3 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 2 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 3 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 2 | 0 | 3 | 2 | 3 | 2 | 3 |
| 4 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| 5 | 0 | 3 | 0 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 2 | 1 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 3 | 0 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 2 | 1 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 3 | 0 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 2 | 1 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 3 | 0 | 2 | 3 | 2 | 3 | 2 |
| 6 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 0 | 3 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |

$b_2 = 1$

| $a_2$ | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| 1 | 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 3 | 2 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 3 | 2 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| 2 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| 3 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 3 | 2 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 3 | 2 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 3 | 2 | 1 | 0 | 1 | 0 | 1 |
| 4 | 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| 5 | 0 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 3 | 2 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 3 | 2 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 3 | 2 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 2 | 3 | 0 | 1 | 0 | 1 | 0 |
| 6 | 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 1 | 0 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |

$b_2 = 2$

| $a_2$ | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| 2 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| 4 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| 5 | 0 | 0 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 1 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 0 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 1 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 0 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 1 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 0 | 2 | 3 | 2 | 3 | 2 | 3 |
| 6 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |

$b_2 = 3$

| $a_2$ | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| 1 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| 2 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| 3 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 4 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| 5 | 0 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| 6 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |

$b_2 = 4$

| $a_2$ | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| 2 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 5 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| 6 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

$b_2 = 5$

| $a_2$ | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| 1 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 4 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| 5 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 6 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |

$b_2 = 6$

| $a_2$ | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| 2 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 5 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| 6 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

**Table 4.** The Grundy values for a $k, l$, 1-bipode $B_{a_1,a_2,b_1,b_2}$.

| $a_2\backslash b_2$ | 0 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 0 | 0 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 1 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 0 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 1 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 0 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 1 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 0 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |

| $a_2\backslash b_2$ | 1 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 3 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 0 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 3 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 0 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |

| $a_2\backslash b_2$ | 2 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

| $a_2\backslash b_2$ | 3 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |

| $a_2\backslash b_2$ | 4 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

| $a_2\backslash b_2$ | 5 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |

| $a_2\backslash b_2$ | 6 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 3 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 4 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 6 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 0 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 1 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 2 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 3 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 4 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | 5 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| | 6 | 2 | 3 | 2 | 3 | 2 | 3 | 2 |
| | $a_1\backslash b_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 3 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | 5 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

**Table 5.** The Grundy values for a $k, l, 2$-bipode $B_{a_1,a_2,b_1,b_2}$.