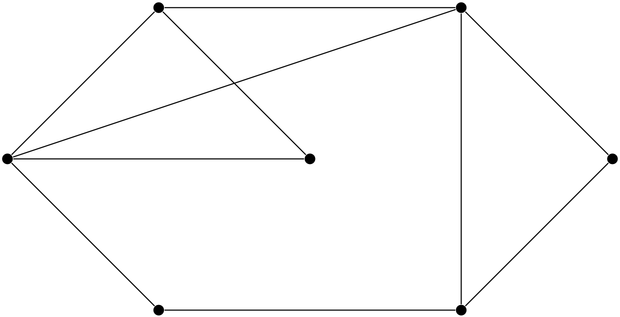
A Vizing-like theorem for union vertex-distinguishing edge coloring

Nicolas Bousquet, Antoine Dailly, Éric Duchêne, Hamamache Kheddouci, Aline Parreau

LIRIS, Université Lyon 1

BGW 2016

Context



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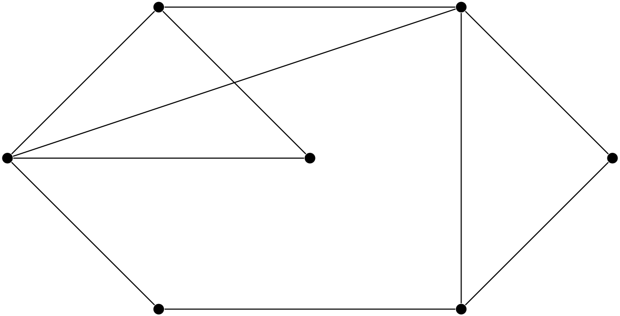

Goal: identify each vertex of the graph by a *code*.

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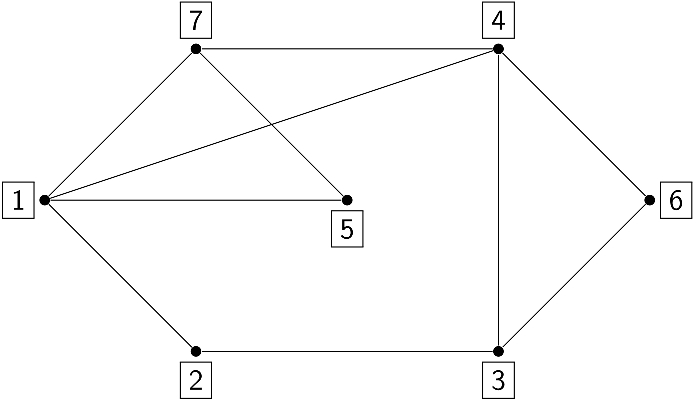

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Codes can be integers

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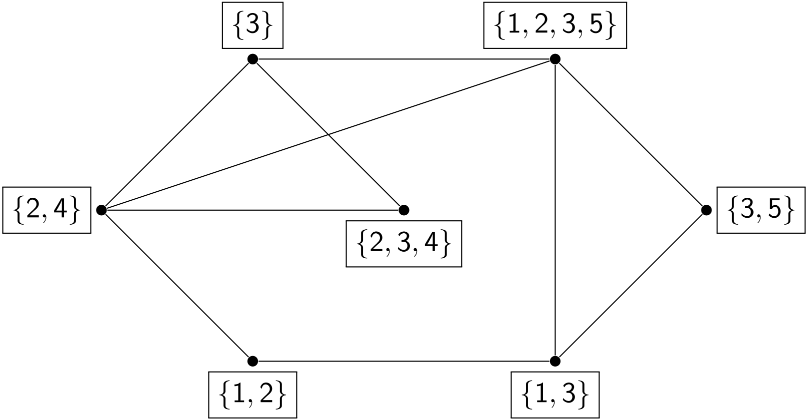

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Codes can be integers, sets of integers

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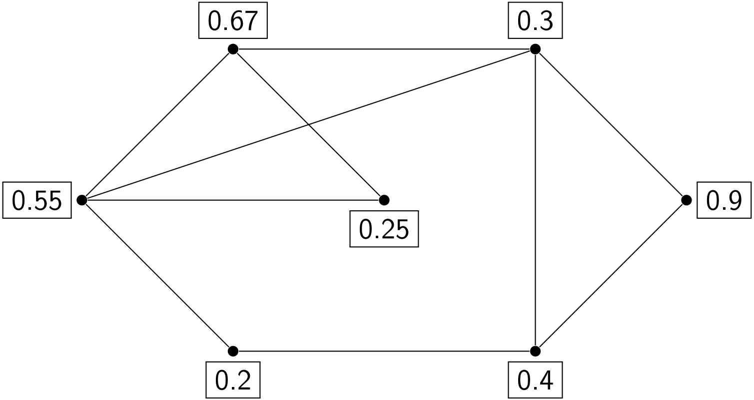

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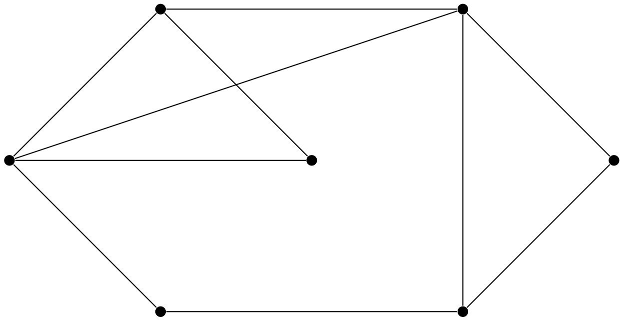
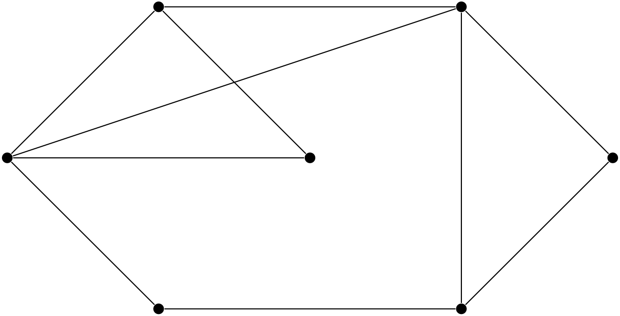


Goal: identify each vertex of the graph by a *code*.
Codes can be integers, sets of integers, floats, ...
⇒ Use of a coloring to generate the codes.

Context

Identifying coloring (Parreau, 2012)



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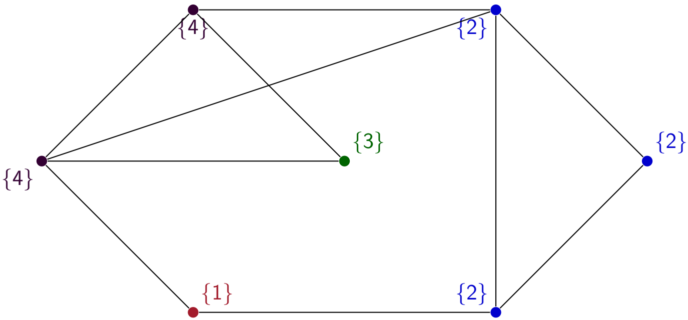

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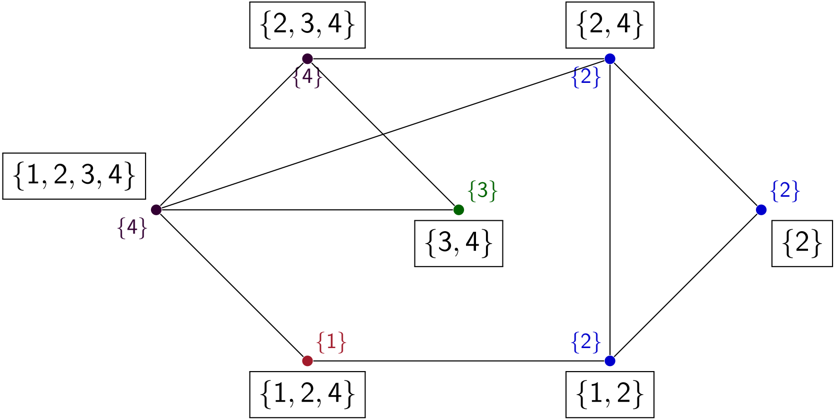
- Vertices are colored by an integer
- Code of a vertex = the union of its and its neighbours' colors



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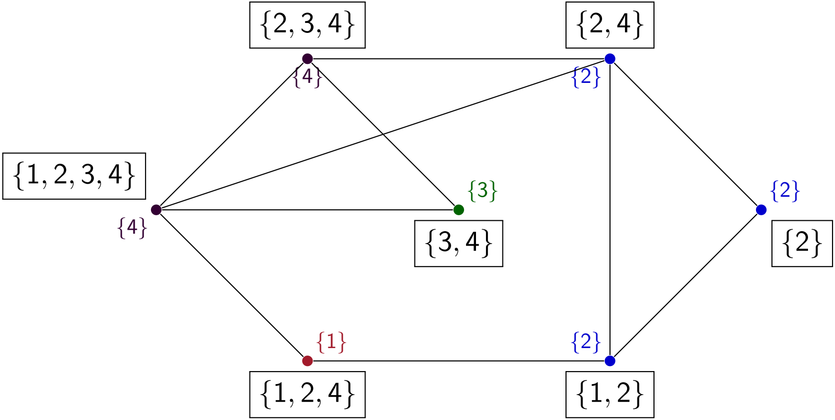
- Minimize the number of colors (parameter χ_{id})



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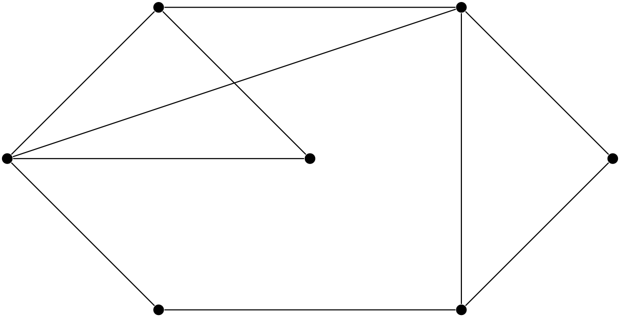
Distinguishing edge coloring (Harary and Plantholt, 1985)



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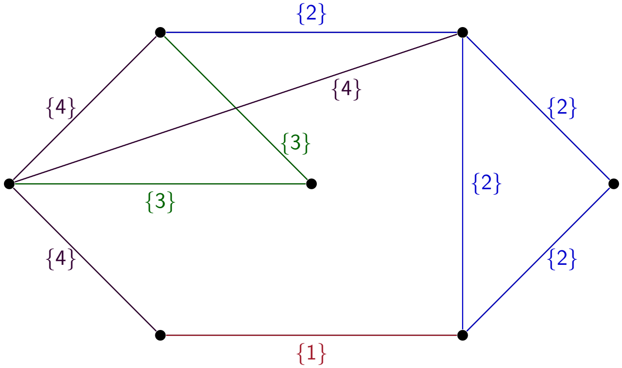

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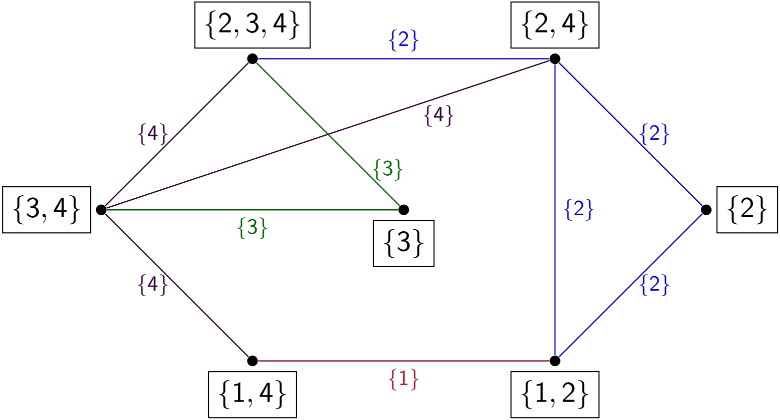

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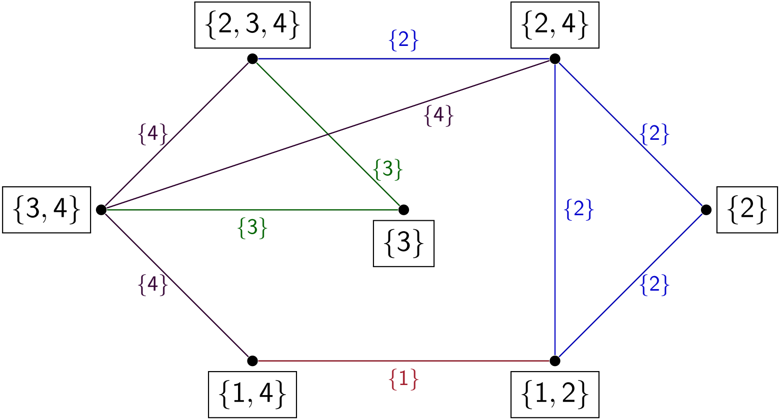

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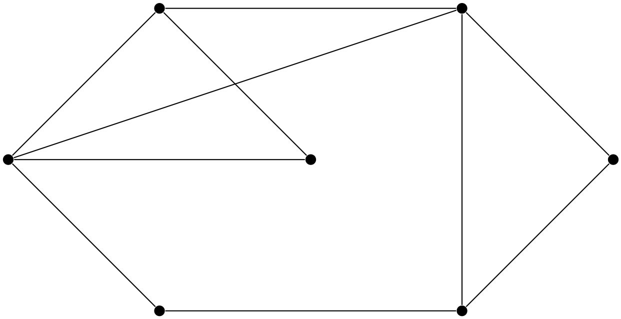

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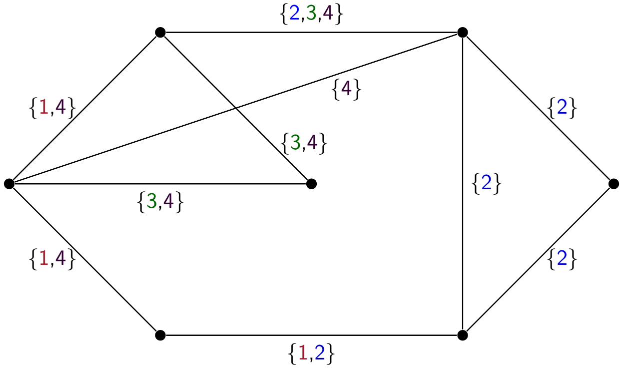

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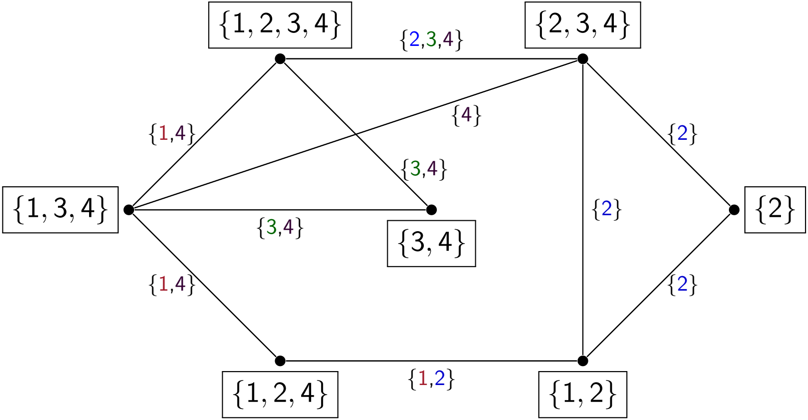

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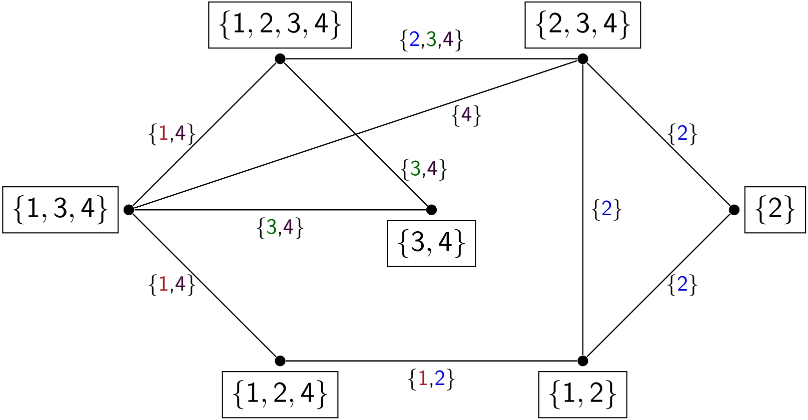

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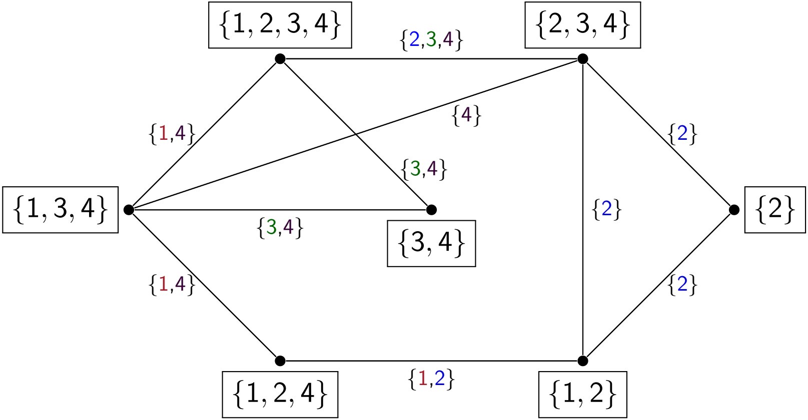

⇒ Only for graphs with connected components of size at least 3 !

Properties

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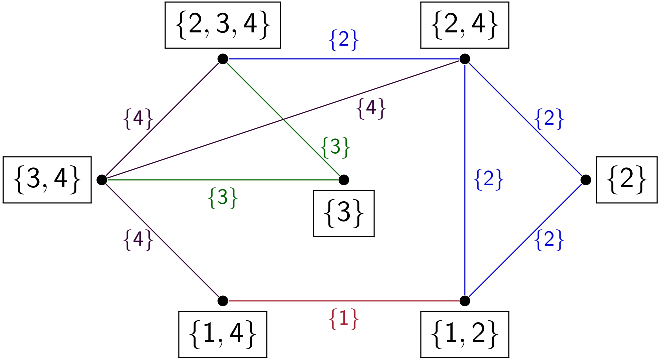
A distinguishing edge coloring is a union-distinguishing edge coloring!



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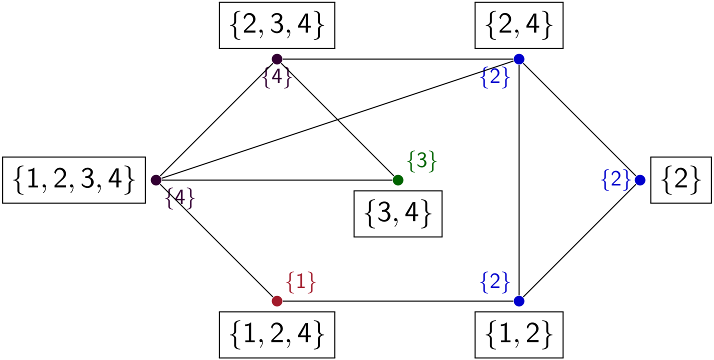
An identifying coloring induces a union-distinguishing edge coloring.



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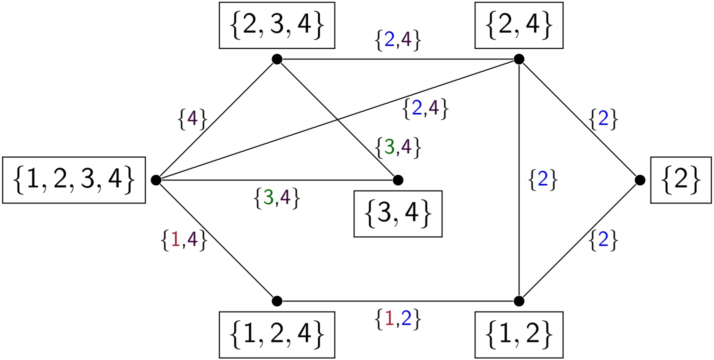

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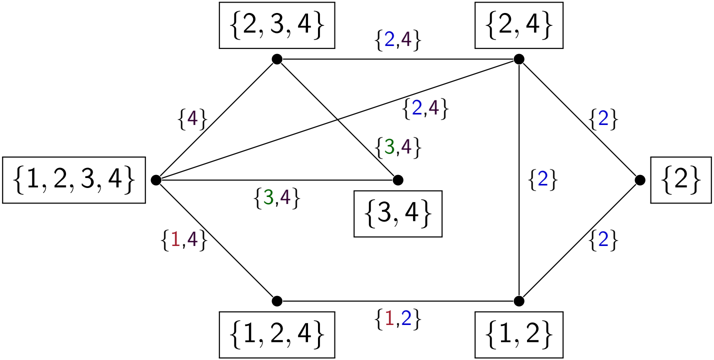

Proposition

For every graph G, we have $\chi_{\cup}(G) \leq min(\chi_S(G), \chi_{id}(G))$.

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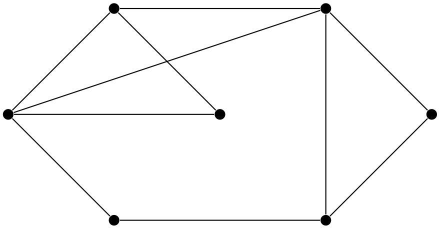
Suppose that we want to construct a union vertex-distinguishing edge coloring of a graph G using k colors.



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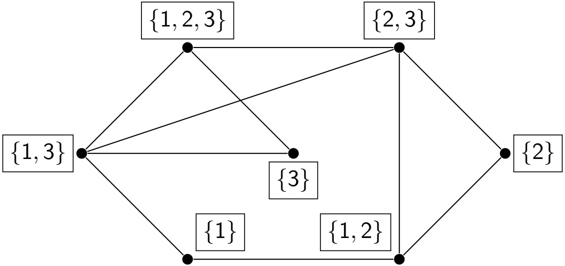
$\{1, \ldots, k\}$ has $2^k - 1$ nonempty subsets, so : $|V(G)| \leq 2^k - 1$.



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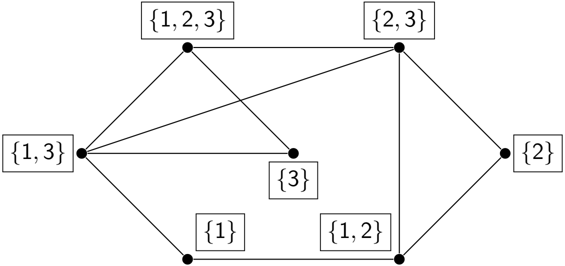

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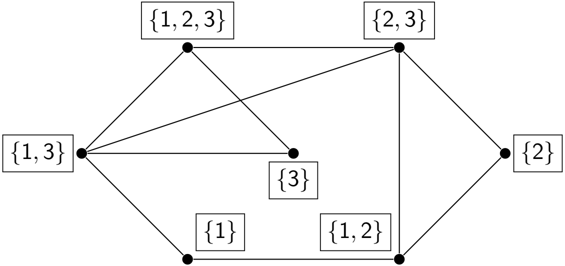

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A graph $G(V, E)$ is said to be optimally colored if $\chi_\cup(G) = \lceil \log_2(|V| + 1) \rceil$.

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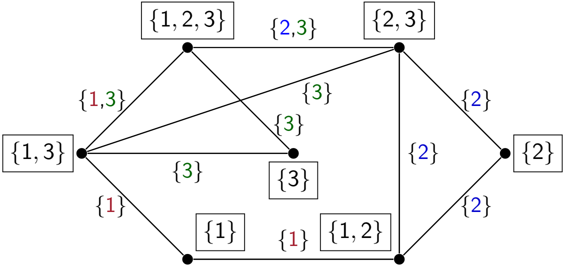

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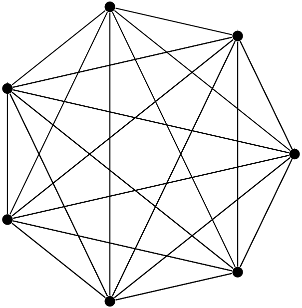

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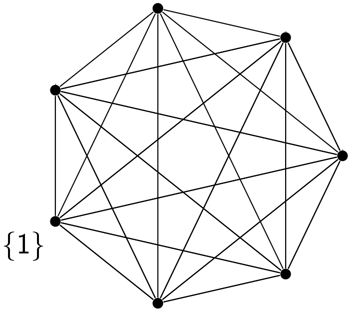

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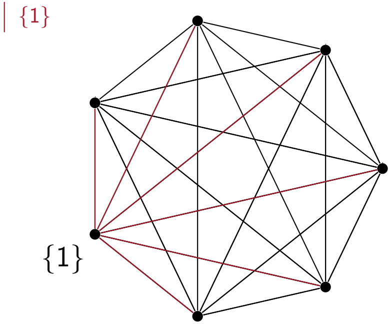

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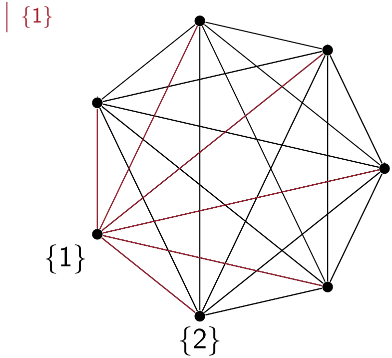

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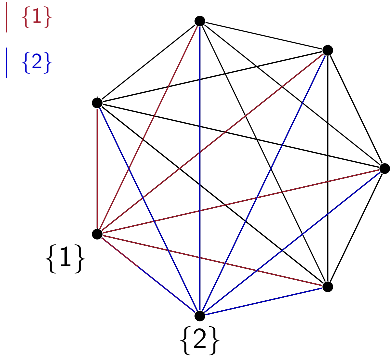

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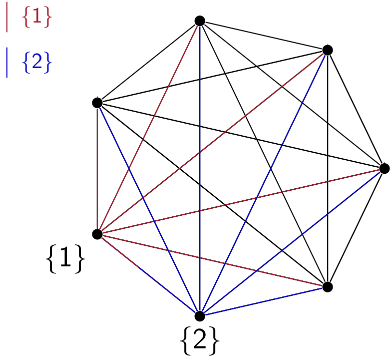

$\Rightarrow$ Only one vertex is identified by a singleton in K_{2^k-1} $\Rightarrow$ K_{2^k-1} needs more than k colors

Results on some classes

Theorem

A path P_n can be optimally colored.

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Induction hypothesis on $n = 2^k + \ell$ ($\ell \leq 2^k - 1$): P_{2^k-1} and P_ℓ can be optimally colored (with conditions on the coloring).

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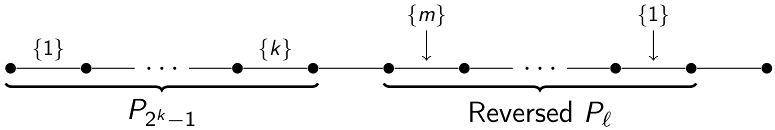

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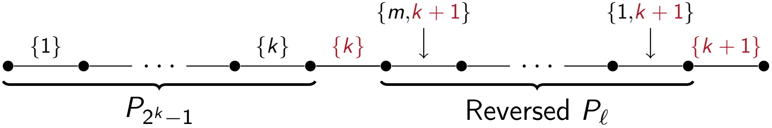

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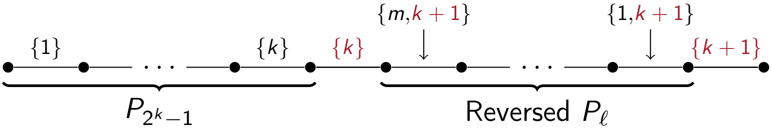

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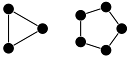
For $n \geq 4$, $n \neq 7$, C_n can be optimally colored.
$\chi_\cup(C_3) = 3$ and $\chi_\cup(C_7) = 4$.



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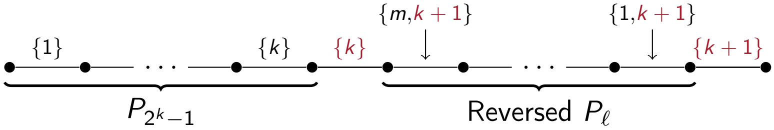

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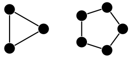

Theorem

A complete binary tree of height at least 1 can be optimally colored.

Upper bound

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For every graph $G(V, E)$, we have $\chi_\cup(G) \leq \lceil \log_2(|V| + 1) \rceil + 2$.

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$\chi_\cup$ can only take three values!

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For every graph $G(V, E)$, we have $\chi_\cup(G) \leq \lceil \log_2(|V| + 1) \rceil + 2$.

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$\chi_\cup(G)$	$\lceil \log_2(\lvert V(G)\rvert + 1) \rceil$	$\lceil \log_2(\lvert V(G)\rvert + 1) \rceil + 1$	$\lceil \log_2(\lvert V(G)\rvert + 1) \rceil + 2$

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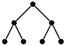
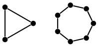
$\chi_{\cup}(G)$	$\lceil \log_2(\lvert V(G)\rvert + 1) \rceil$	$\lceil \log_2(\lvert V(G)\rvert + 1) \rceil + 1$	$\lceil \log_2(\lvert V(G)\rvert + 1) \rceil + 2$
	Paths  Cycles  Complete binary trees 	K_{2^k-1}  C_3, C_7 	

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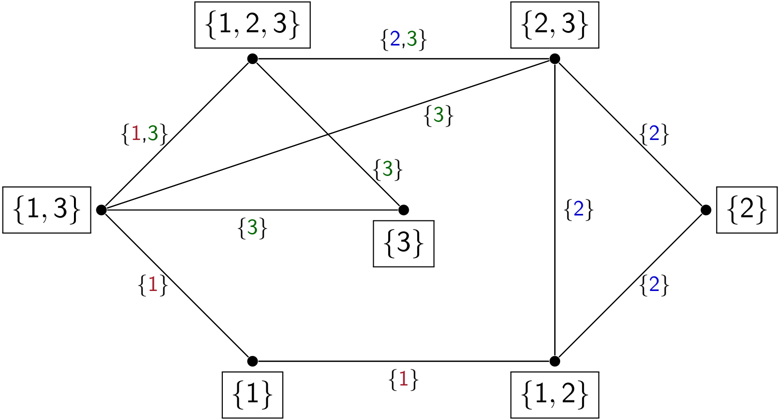

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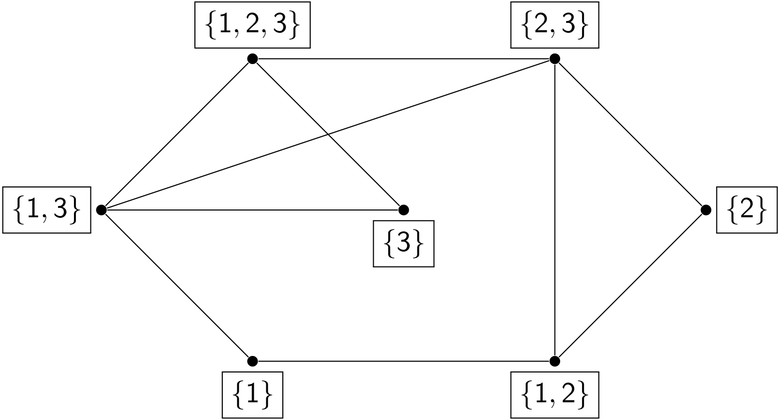

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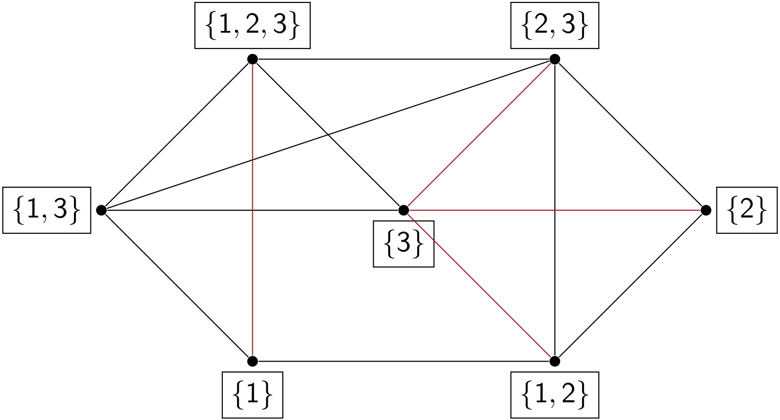

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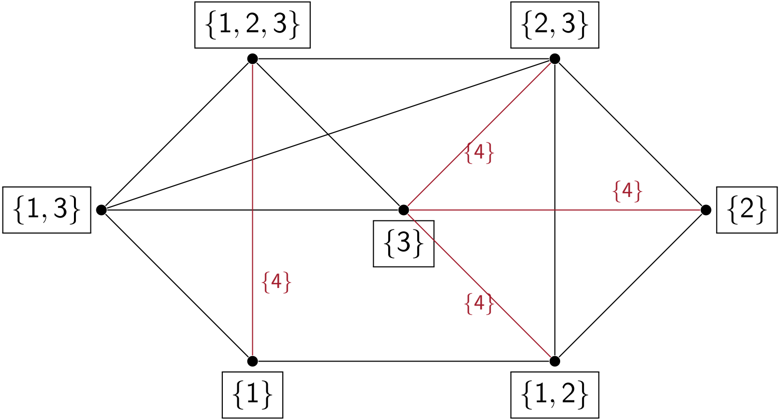

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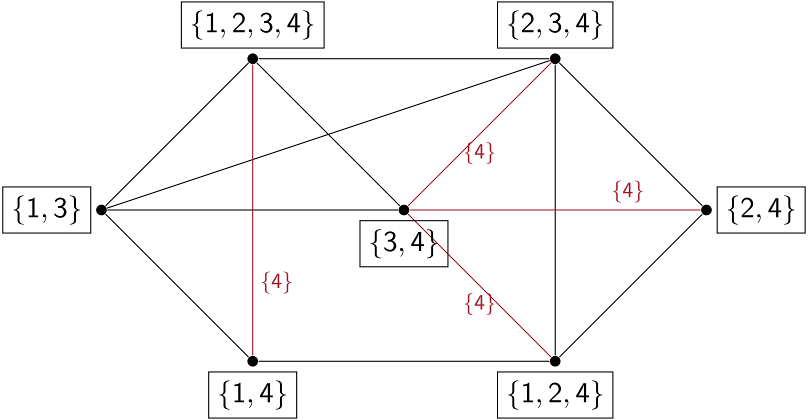
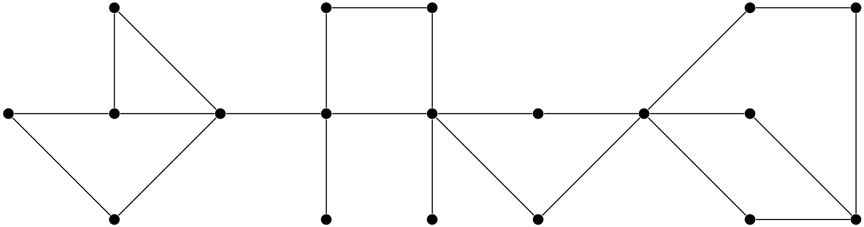


Proof

From a graph G :



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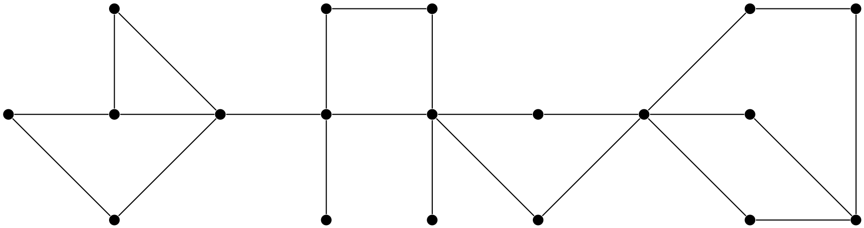
1. Extract H, forest of stars subdivided at most once and edge-subgraph of G;



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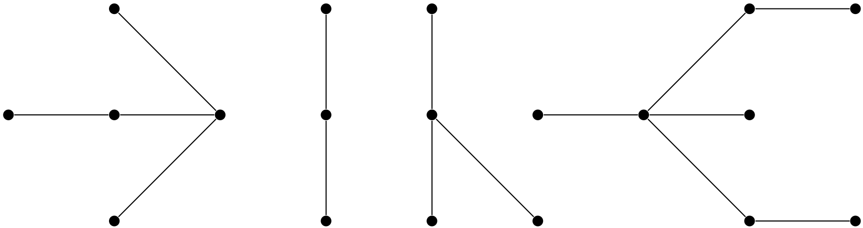
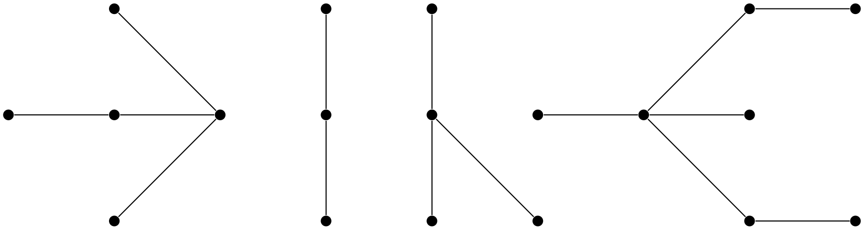


Proof

From a graph G :

1. Extract H, forest of stars subdivided at most once and edge-subgraph of G;
2. Optimally color each component of H;



Proof

From a graph G :

1. Extract H, forest of stars subdivided at most once and edge-subgraph of G;
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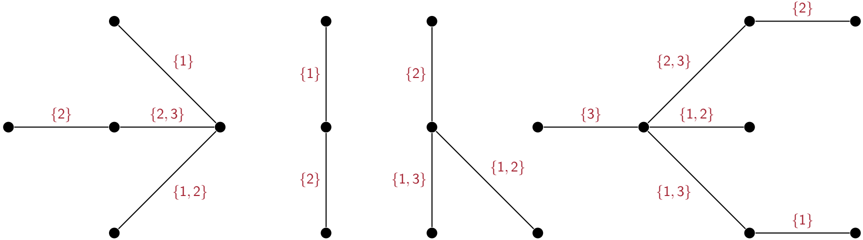

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1. Extract H, forest of stars subdivided at most once and edge-subgraph of G;
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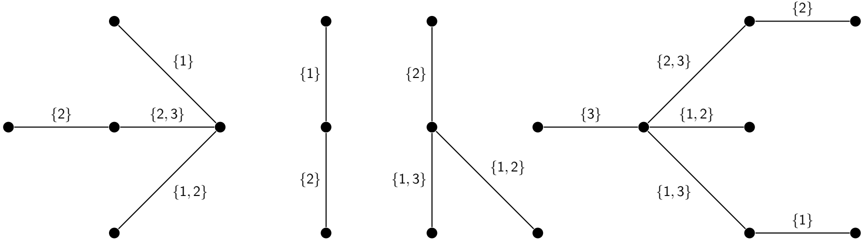
3. Color their disjoint union, H, with the optimal number of colors plus one;



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	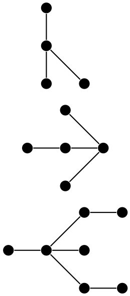		

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	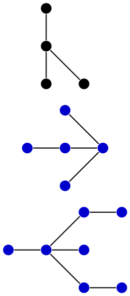		

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	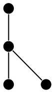	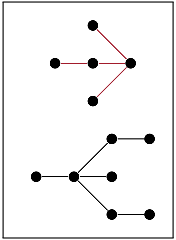	

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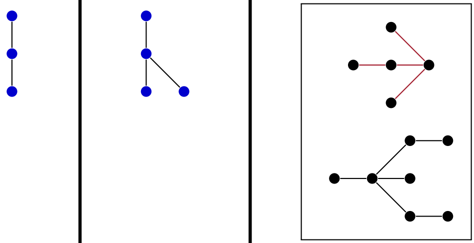

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		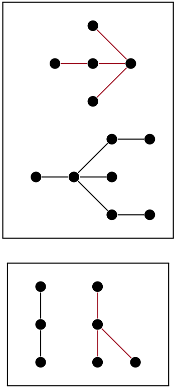	

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			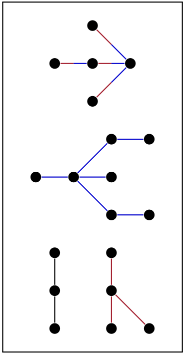

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Color 5 added

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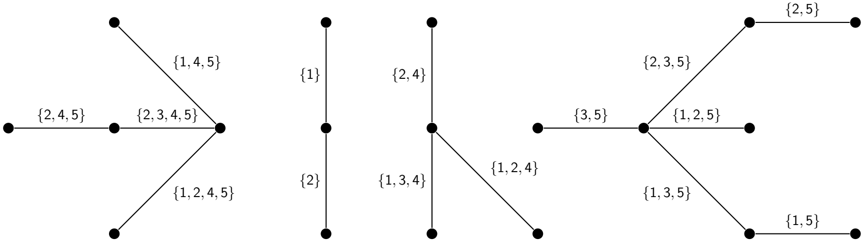

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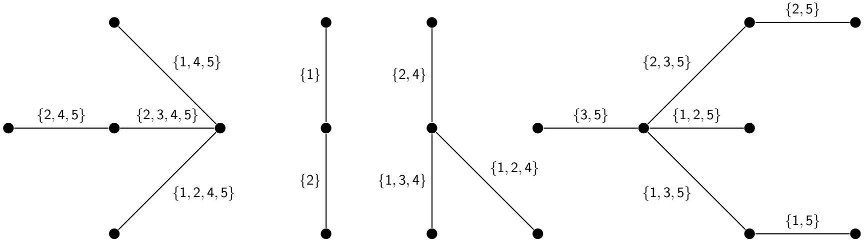

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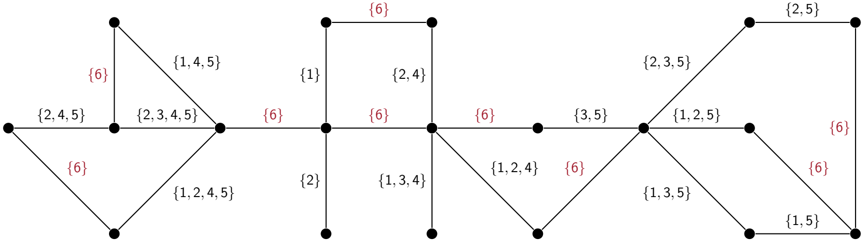

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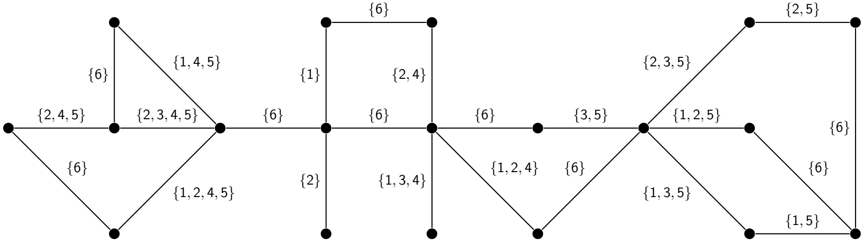

Summary

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For every graph $G(V, E)$, we have
$\lceil \log_2(|V| + 1) \rceil \leq \chi_\cup(G) \leq \lceil \log_2(|V| + 1) \rceil + 2$.

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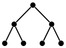
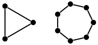
	Paths  Cycles  Complete binary trees 	K_{2^k-1}  C_3, C_7 	

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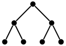
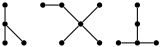
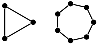
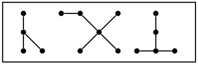
	Paths  Cycles  Complete binary trees  Stars subdivided at most once 	K_{2^k-1}  C_3, C_7  Forest of stars subdivided at most once 	

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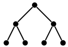
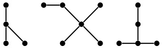
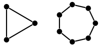
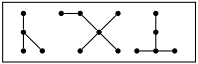
	Paths  Cycles  Complete binary trees  Stars subdivided at most once 	K_{2^k-1}  C_3, C_7  Forest of stars subdivided at most once  Hamiltonian graphs; Edge-supergraphs of stars subdivided at most once	

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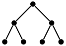
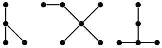
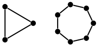
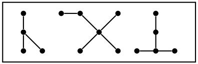
	Paths  Cycles  Complete binary trees  Stars subdivided at most once 	K_{2^k-1}  C_3, C_7  Forest of stars subdivided at most once  Hamiltonian graphs Edge-supergraphs of stars subdivided at most once	?

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