Octal Games on Graphs

Laurent Beaudou[1], Pierre Coupechoux[2], Antoine Dailly[3],
Sylvain Gravier[4], Julien Moncel[2], Aline Parreau[3], Éric Sopena[5]

[1]LIMOS, Clermont-Ferrand
[2]LAAS, Toulouse
[3]LIRIS, Lyon
[4]Institut Fourier, Grenoble
[5]LaBRI, Bordeaux

This work is part of the ANR GAG (Graphs and Games).

CGTC 2017

Octal games

Definition

Octal games are:

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137 I I I I I

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137 

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137 

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137
- CRAM on a single row is 0.07 

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137
- CRAM on a single row is 0.07 

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137
- CRAM on a single row is 0.07 

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137
- CRAM on a single row is 0.07
- The JAMES BOND GAME is 0.007 

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137
- CRAM on a single row is 0.07
- The JAMES BOND GAME is 0.007 

Octal games

Definition

Octal games are:

- **impartial** games;
- played on **heaps of counters**;
- whose rules are defined by an **octal code**.

Examples

- NIM is 0.3333...
- KAYLES is 0.137
- CRAM on a single row is 0.07
- The JAMES BOND GAME is 0.007 

Octal games

Grundy sequence

The **Grundy sequence** of an octal game is the sequence of its Grundy values for heaps of size $0, 1, 2, \ldots$

Octal games

Grundy sequence

The **Grundy sequence** of an octal game is the sequence of its Grundy values for heaps of size $0, 1, 2, \ldots$

Examples

- NIM: 0,1,2,3,4,5,...

Octal games

Grundy sequence

The **Grundy sequence** of an octal game is the sequence of its Grundy values for heaps of size $0, 1, 2, \ldots$

Examples

- NIM: 0,1,2,3,4,5,. . .
- KAYLES: 0,1,2,3,1,4,3,2,. . . after a pre-period 72 it becomes periodic with period 12;

Octal games

Grundy sequence

The **Grundy sequence** of an octal game is the sequence of its Grundy values for heaps of size $0, 1, 2, \ldots$

Examples

- NIM: 0,1,2,3,4,5,...
- KAYLES: 0,1,2,3,1,4,3,2,... after a pre-period 72 it becomes periodic with period 12;
- CRAM on a single row: 0,1,1,2,0,3,1,1,... after a pre-period 53 it becomes periodic with period 34

Octal games

Grundy sequence

The **Grundy sequence** of an octal game is the sequence of its Grundy values for heaps of size $0, 1, 2, \ldots$

Examples

- NIM: 0,1,2,3,4,5,...
- KAYLES: 0,1,2,3,1,4,3,2,... after a pre-period 72 it becomes periodic with period 12;
- CRAM on a single row: 0,1,1,2,0,3,1,1,... after a pre-period 53 it becomes periodic with period 34
- The JAMES BOND GAME: 0,0,0,1,1,1,2,2,0,3,3,1,1,1,0,4,... still open, 2^{28} values computed!

Octal games

Grundy sequence

The **Grundy sequence** of an octal game is the sequence of its Grundy values for heaps of size $0, 1, 2, \ldots$

Examples

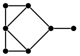
- NIM: 0,1,2,3,4,5,...
- KAYLES: 0,1,2,3,1,4,3,2,... after a pre-period 72 it becomes periodic with period 12;
- CRAM on a single row: 0,1,1,2,0,3,1,1,... after a pre-period 53 it becomes periodic with period 34
- The JAMES BOND GAME: 0,0,0,1,1,1,2,2,0,3,3,1,1,1,0,4,... still open, 2^{28} values computed!

Conjecture (Guy)

All **finite** octal games have **ultimately periodic** Grundy sequences.

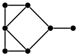
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
	

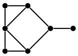
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	
	

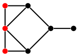
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	
	

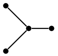
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
	

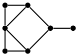
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
	

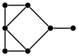
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
| | | | | | |	
Splitting a heap	

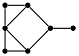
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
	
Splitting a heap	

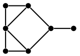
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
| | | |	
Splitting a heap	

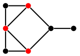
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
| | | |	
Splitting a heap	**Disconnecting** a graph

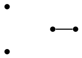
Octal games on graphs

Natural generalization of the definition:

<table>
<tr><th>Playing on heaps</th><th>Playing on graphs</th></tr>
<tr><td>Removing counters from a heap</td><td>Removing connected vertices from a graph</td></tr>
<tr><td>| | | |</td><td></td></tr>
<tr><td>Splitting a heap</td><td>Disconnecting a graph</td></tr>
</table>

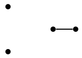
Octal games on graphs

Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
	
Splitting a heap	**Disconnecting** a graph

Octal games on graphs

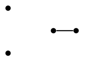
Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
	
Splitting a heap	**Disconnecting** a graph

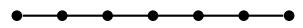
Playing on a heap

Octal games on graphs

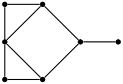
Natural generalization of the definition:

Playing on heaps	Playing on graphs
Removing **counters** from a **heap**	Removing **connected vertices** from a **graph**
	
Splitting a heap	**Disconnecting** a graph

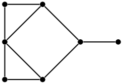
Playing on a heap $\equiv$ Playing on a path



Octal games on graphs

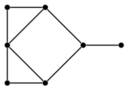


Octal games on graphs



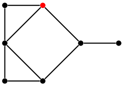
- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)

Octal games on graphs



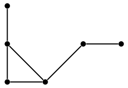
- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6

Octal games on graphs



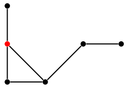
- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6

Octal games on graphs



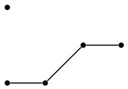
- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6

Octal games on graphs



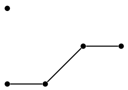
- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6

Octal games on graphs



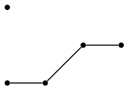
- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6

Octal games on graphs



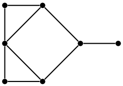
- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6
  - Study of cycles, wheels, random graphs, . . .

Octal games on graphs



- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6
  - Study of cycles, wheels, random graphs, ...
- Scoring version of 0.6 (Duchêne *et al.*, 2017+)

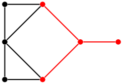
Octal games on graphs



- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6
  - Study of cycles, wheels, random graphs, ...
- Scoring version of 0.6 (Duchêne *et al.*, 2017+)

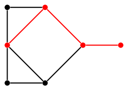
- NODE-KAYLES is **not** an octal game

Octal games on graphs



- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6
  - Study of cycles, wheels, random graphs, ...
- Scoring version of 0.6 (Duchêne *et al.*, 2017+)
- NODE-KAYLES is **not** an octal game

Octal games on graphs

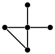


- ARC-KAYLES (Schaeffer, 1978) is 0.07
  - *FPT* when parameterized by the number of rounds played (Lampis & Mitsou, 2014)
  - Study of cycles and wheels, some sort of periodicity on specific stars (Huggan & Stevens, 2016)
- GRIM (Adams *et al.*, 2016) is 0.6
  - Study of cycles, wheels, random graphs, . . .
- Scoring version of 0.6 (Duchêne *et al.*, 2017+)
- NODE-KAYLES is **not** an octal game

The game 0.33 on graphs

Rules

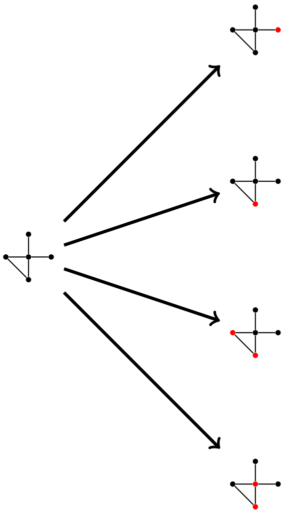
In the game 0.33, both players alternate removing **one or two adjacent** vertices **without disconnecting** the graph.



The game 0.33 on graphs

Rules

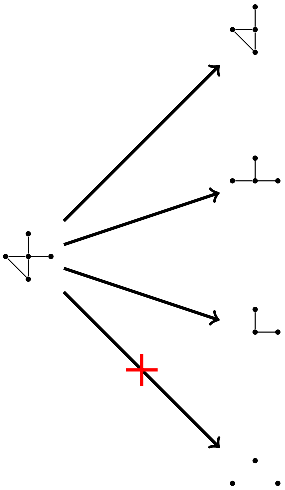
In the game 0.33, both players alternate removing **one or two adjacent** vertices **without disconnecting** the graph.



The game 0.33 on graphs

Rules

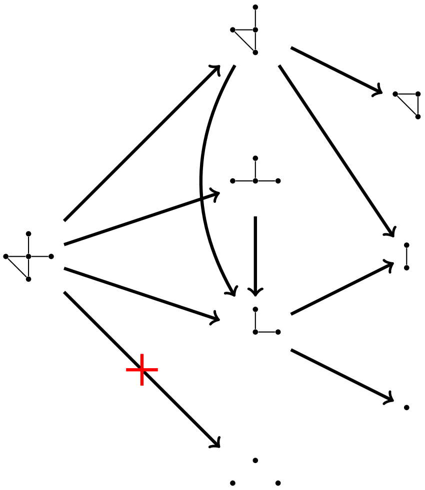
In the game 0.33, both players alternate removing **one or two adjacent** vertices **without disconnecting** the graph.



The game 0.33 on graphs

Rules

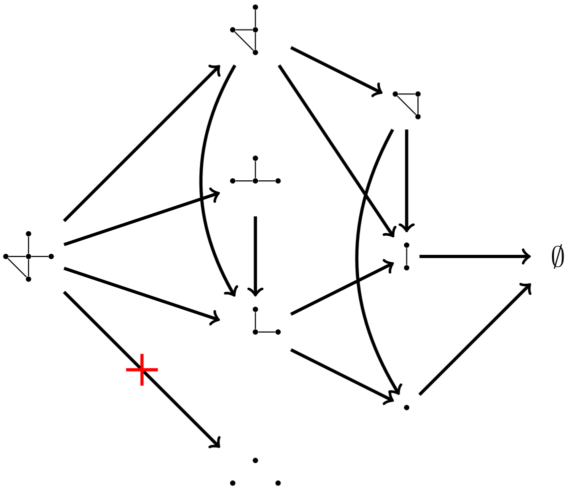
In the game 0.33, both players alternate removing **one or two adjacent** vertices **without disconnecting** the graph.



The game 0.33 on graphs

Rules

In the game 0.33, both players alternate removing **one or two adjacent** vertices **without disconnecting** the graph.



The game 0.33 on graphs

Rules

In the game 0.33, both players alternate removing **one or two adjacent** vertices **without disconnecting** the graph.

Remark

For every integer n, we have $\mathcal{G}(P_n) = n \bmod 3$.

The game 0.33 on graphs

Rules

In the game 0.33, both players alternate removing **one or two adjacent** vertices **without disconnecting** the graph.

Remark

For every integer n, we have $\mathcal{G}(P_n) = n \bmod 3$.

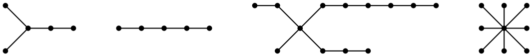
Corollary

A path can be **reduced** to its **length modulo 3** without changing its Grundy value.

The game 0.33 on subdivided stars

Subdivided stars

A **subdivided star** $S_{\ell_1,\dots,\ell_k}$ is a graph composed of a central vertex connected to k paths of length $\ell_1, \dots, \ell_k$.

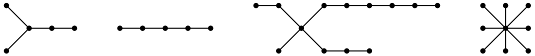


$S_{1,1,2}$ S_4 $S_{1,2,3,6}$ $S_{1,1,1,1,1,1,1,1}$

The game 0.33 on subdivided stars

Subdivided stars

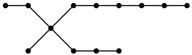
A **subdivided star** $S_{\ell_1,\ldots,\ell_k}$ is a graph composed of a central vertex connected to k paths of length $\ell_1, \ldots, \ell_k$.



$S_{1,1,2}$ S_4 $S_{1,2,3,6}$ $S_{1,1,1,1,1,1,1,1}$

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.
In other words, each path of a subdivided star can be **reduced** to its **length modulo 3** without changing the Grundy value of the star.

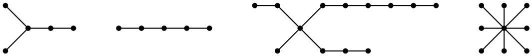


$S_{1,2,3,6}$

The game 0.33 on subdivided stars

Subdivided stars

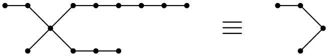
A **subdivided star** $S_{\ell_1,\ldots,\ell_k}$ is a graph composed of a central vertex connected to k paths of length $\ell_1, \ldots, \ell_k$.



$S_{1,1,2}$ S_4 $S_{1,2,3,6}$ $S_{1,1,1,1,1,1,1,1}$

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.
In other words, each path of a subdivided star can be **reduced** to its **length modulo 3** without changing the Grundy value of the star.

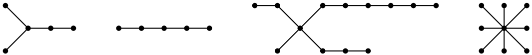


$S_{1,2,3,6}$ $S_{1,2} = P_4$

The game 0.33 on subdivided stars

Subdivided stars

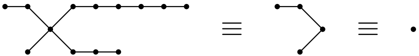
A **subdivided star** $S_{\ell_1,\dots,\ell_k}$ is a graph composed of a central vertex connected to k paths of length $\ell_1, \dots, \ell_k$.



$S_{1,1,2}$ S_4 $S_{1,2,3,6}$ $S_{1,1,1,1,1,1,1,1}$

Theorem

For all $\ell_1, \dots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\dots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\dots,\ell_k \bmod 3})$.
In other words, each path of a subdivided star can be **reduced** to its **length modulo 3** without changing the Grundy value of the star.



$S_{1,2,3,6}$ $S_{1,2} = P_4$ P_1

The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.

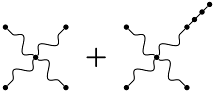
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.



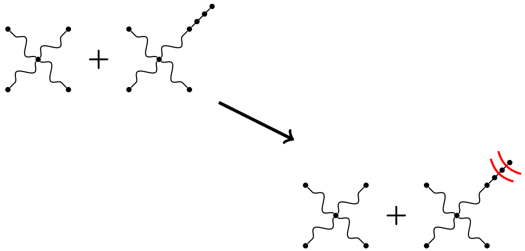
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.



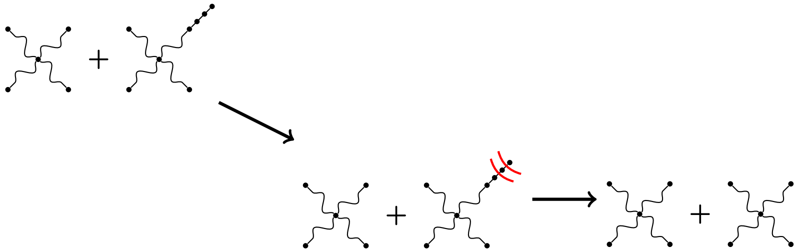
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.



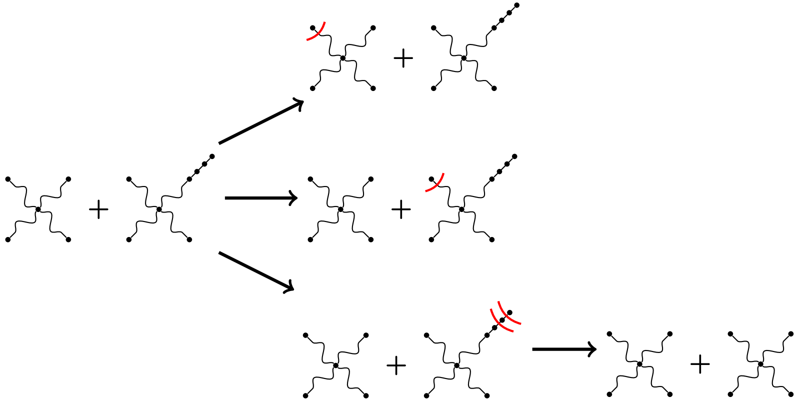
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.



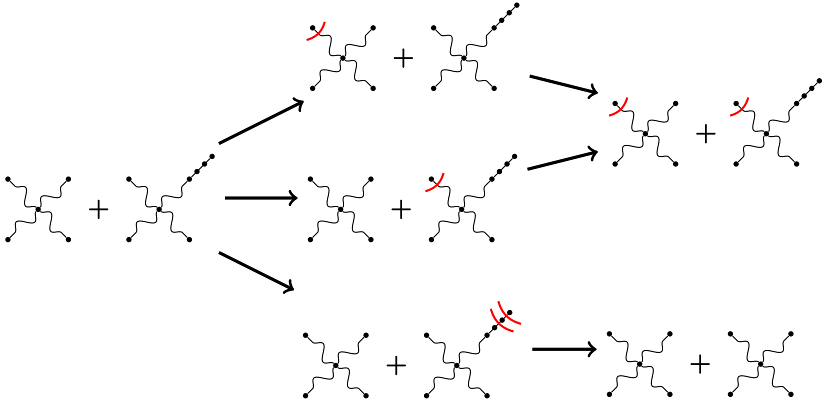
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \dots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\dots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\dots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\dots,\ell_i,\dots,\ell_k}) = \mathcal{G}(S_{\ell_1,\dots,\ell_i+3,\dots,\ell_k})$.



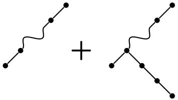
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.



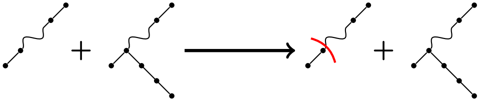
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \text{ mod } 3,\ldots,\ell_k \text{ mod } 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.



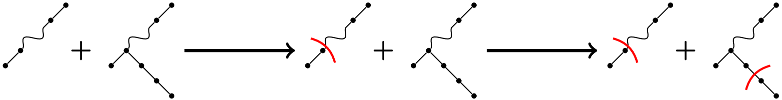
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.



$P_\ell + S_{1,1,\ell}$

The game 0.33 on subdivided stars

Lemma

For all ℓ, we have $\mathcal{G}(S_{1,1,\ell}) = \ell \bmod 3$.

The game 0.33 on subdivided stars

Lemma

For all ℓ, we have $\mathcal{G}(S_{1,1,\ell}) = \ell \bmod 3$.

Proof

We use induction on ℓ.

$\mathcal{G}($  $) = 0$ $\mathcal{G}($  $) = 1$

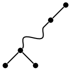
The game 0.33 on subdivided stars

Lemma

For all ℓ, we have $\mathcal{G}(S_{1,1,\ell}) = \ell \bmod 3$.

Proof

We use induction on ℓ.



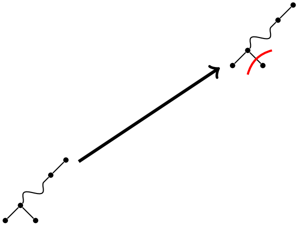
The game 0.33 on subdivided stars

Lemma

For all ℓ, we have $\mathcal{G}(S_{1,1,\ell}) = \ell \bmod 3$.

Proof

We use induction on ℓ.



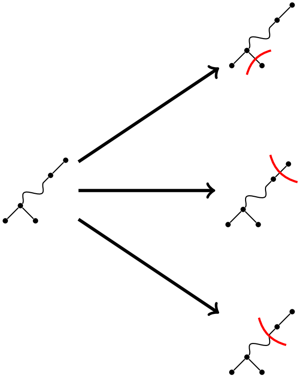
The game 0.33 on subdivided stars

Lemma

For all ℓ, we have $\mathcal{G}(S_{1,1,\ell}) = \ell \bmod 3$.

Proof

We use induction on ℓ.



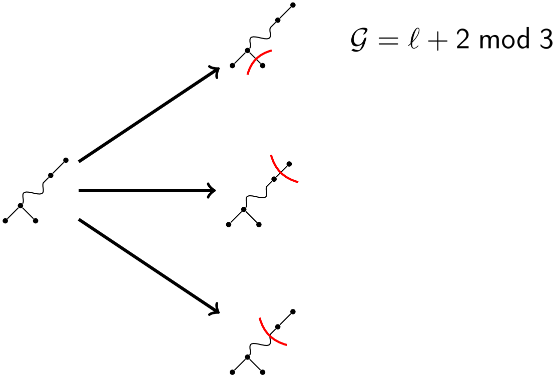
The game 0.33 on subdivided stars

Lemma

For all ℓ, we have $\mathcal{G}(S_{1,1,\ell}) = \ell \bmod 3$.

Proof

We use induction on ℓ.



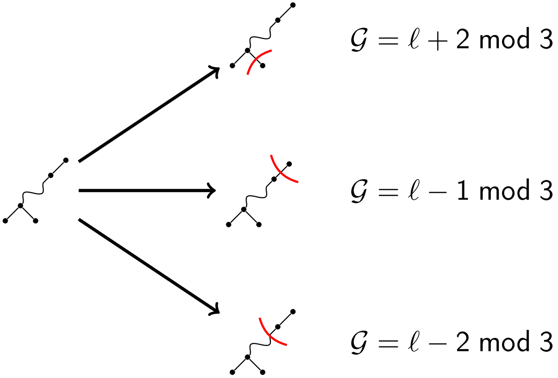
The game 0.33 on subdivided stars

Lemma

For all ℓ, we have $\mathcal{G}(S_{1,1,\ell}) = \ell \bmod 3$.

Proof

We use induction on ℓ.



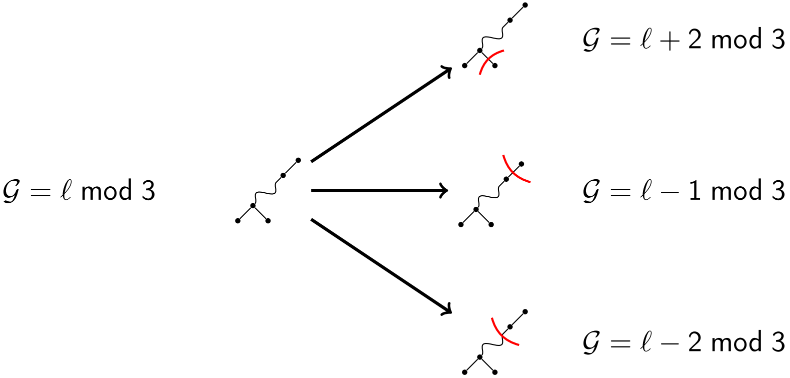
The game 0.33 on subdivided stars

Lemma

For all ℓ, we have $\mathcal{G}(S_{1,1,\ell}) = \ell \bmod 3$.

Proof

We use induction on ℓ.



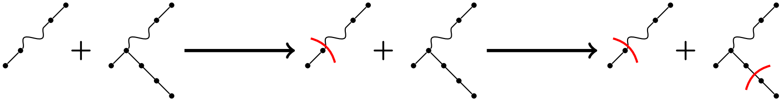
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.



$$P_\ell + S_{1,1,\ell}$$

$$\mathcal{G}(P_\ell + S_{1,1,\ell}) = 0$$

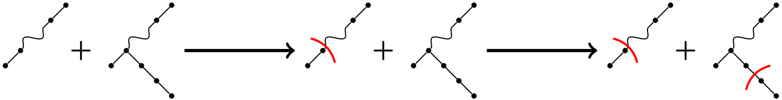
The game 0.33 on subdivided stars

Theorem

For all $\ell_1, \ldots, \ell_k$, we have $\mathcal{G}(S_{\ell_1,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1 \bmod 3,\ldots,\ell_k \bmod 3})$.

Proof

We prove by induction that $\mathcal{G}(S_{\ell_1,\ldots,\ell_i,\ldots,\ell_k}) = \mathcal{G}(S_{\ell_1,\ldots,\ell_i+3,\ldots,\ell_k})$.

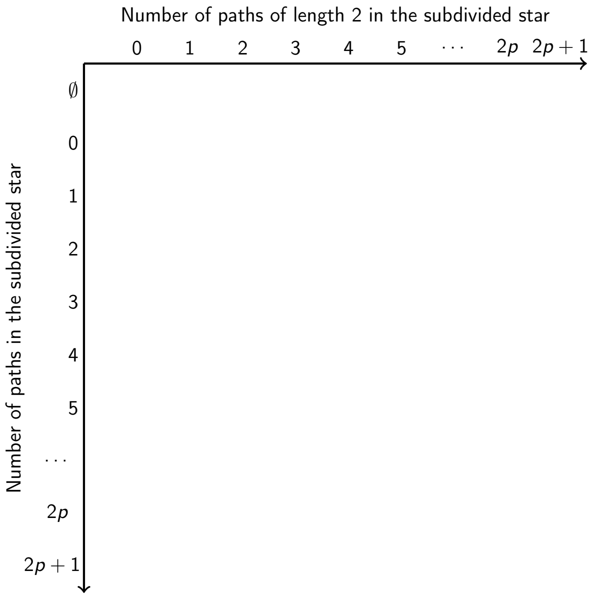


$$P_\ell + S_{1,1,\ell}$$

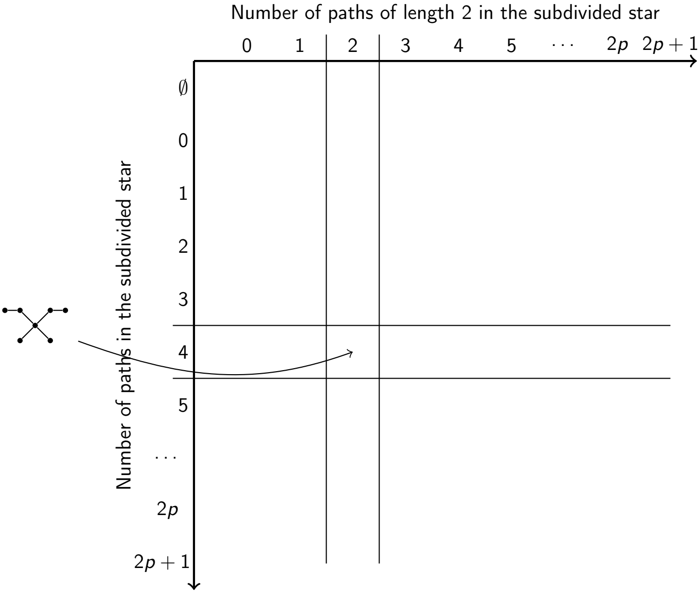
$$\mathcal{G}(P_\ell + S_{1,1,\ell}) = 0$$

$\Rightarrow$ We only need to study stars with paths of length 1 and 2

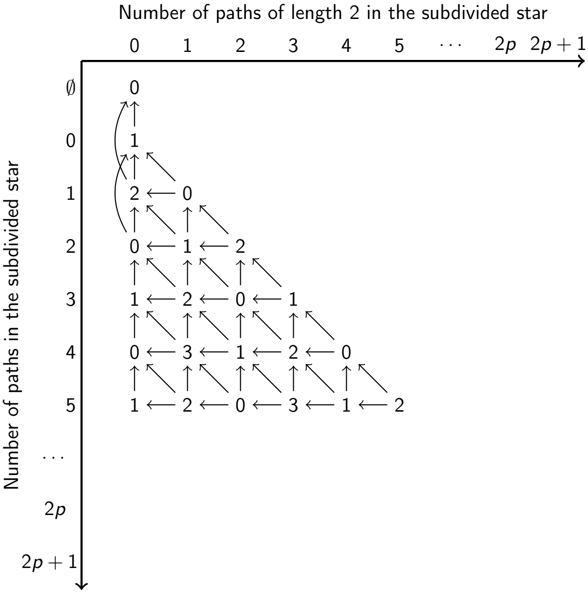
Grundy values of subdivided stars for the game 0.33



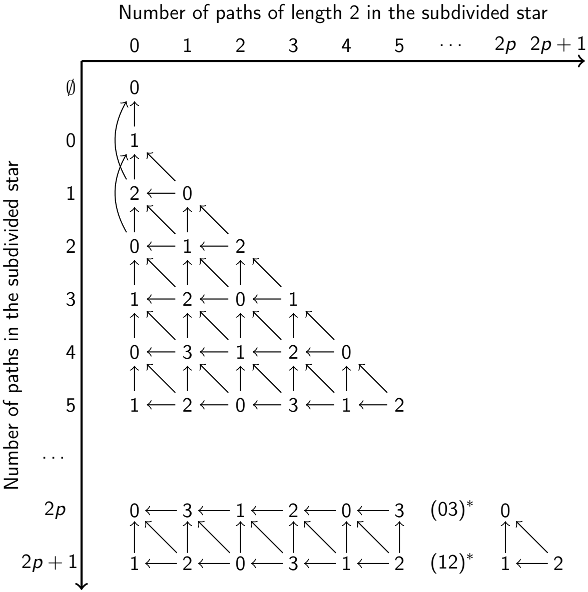
Grundy values of subdivided stars for the game 0.33



Grundy values of subdivided stars for the game 0.33



Grundy values of subdivided stars for the game 0.33



The game 0.33 on subdivided bistars

Subdivided bistars

The **subdivided bistar** $S_1 \stackrel{m}{\bullet\!-\!\bullet} S_2$ is the graph constructed by joining the central vertices of two subdivided stars S_1 and S_2 by a path of m edges.

$S_{1,1} \stackrel{1}{\bullet\!-\!\bullet} S_{1,1}$ $S_{1,2} \stackrel{2}{\bullet\!-\!\bullet} \emptyset$ $S_{1,2,3} \stackrel{3}{\bullet\!-\!\bullet} S_{2,4}$

The game 0.33 on subdivided bistars

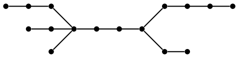
Subdivided bistars

The **subdivided bistar** $S_1 \overset{m}{\bullet\!\!-\!\!\bullet} S_2$ is the graph constructed by joining the central vertices of two subdivided stars S_1 and S_2 by a path of m edges.

$S_{1,1} \overset{1}{\bullet\!\!-\!\!\bullet} S_{1,1}$ $S_{1,2} \overset{2}{\bullet\!\!-\!\!\bullet} \emptyset$ $S_{1,2,3} \overset{3}{\bullet\!\!-\!\!\bullet} S_{2,4}$

Theorem

Each path of a subdivided bistar can be **reduced** to its **length modulo 3** without changing the Grundy value of the bistar.



$S_{1,2,3} \overset{3}{\bullet\!\!-\!\!\bullet} S_{2,4}$

The game 0.33 on subdivided bistars

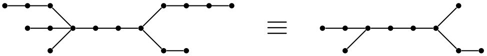
Subdivided bistars

The **subdivided bistar** $S_1 \stackrel{m}{\bullet\!-\!\bullet} S_2$ is the graph constructed by joining the central vertices of two subdivided stars S_1 and S_2 by a path of m edges.

$S_{1,1} \stackrel{1}{\bullet\!-\!\bullet} S_{1,1}$ $S_{1,2} \stackrel{2}{\bullet\!-\!\bullet} \emptyset$ $S_{1,2,3} \stackrel{3}{\bullet\!-\!\bullet} S_{2,4}$

Theorem

Each path of a subdivided bistar can be **reduced** to its **length modulo 3** without changing the Grundy value of the bistar.



$S_{1,2,3} \stackrel{3}{\bullet\!-\!\bullet} S_{2,4} \equiv S_{1,2} \stackrel{3}{\bullet\!-\!\bullet} S_{1,2}$

The game 0.33 on subdivided bistars

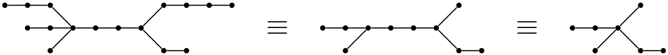
Subdivided bistars

The **subdivided bistar** $S_1 \overset{m}{\bullet\!-\!\bullet} S_2$ is the graph constructed by joining the central vertices of two subdivided stars S_1 and S_2 by a path of m edges.

$S_{1,1} \overset{1}{\bullet\!-\!\bullet} S_{1,1}$ $S_{1,2} \overset{2}{\bullet\!-\!\bullet} \emptyset$ $S_{1,2,3} \overset{3}{\bullet\!-\!\bullet} S_{2,4}$

Theorem

Each path of a subdivided bistar can be **reduced** to its **length modulo 3** without changing the Grundy value of the bistar.



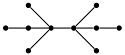
$S_{1,2,3} \overset{3}{\bullet\!-\!\bullet} S_{2,4} \equiv S_{1,2} \overset{3}{\bullet\!-\!\bullet} S_{1,2} \equiv S_{1,1,2,2}$

The game 0.33 on subdivided bistars

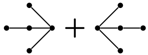
We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.

The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.



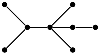
Playing on a subdivided bistar



Playing independently on the two subdivided stars

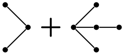
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.



Playing on a subdivided bistar

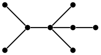
$\sim ?$



Playing independently on the two subdivided stars

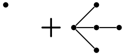
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.



Playing on a subdivided bistar

?

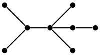


Playing independently on the two subdivided stars

. . . except at the end!

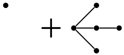
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.



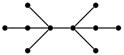
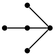
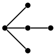
Playing on a subdivided bistar

?



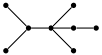
Playing independently on the two subdivided stars

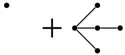
. . . except at the end!

$\mathcal{G}($  $) = 0$ $\mathcal{G}($  $+$  $) = 0$

The game 0.33 on subdivided bistars

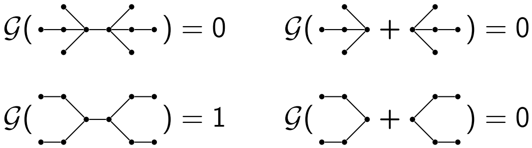
We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.





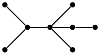
Playing on a subdivided bistar ? Playing independently on the two subdivided stars

…except at the end!



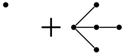
The game 0.33 on subdivided bistars

We want to **directly** compute the Grundy value of a subdivided bistar by using the Grundy values of its stars.



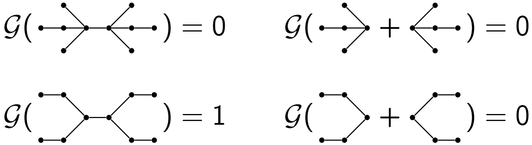
Playing on a subdivided bistar

?



Playing independently on the two subdivided stars

...except at the end!



$\Rightarrow$ **Refinement** of $\equiv$

Refinement of $\equiv$ for subdivided bistars

Reminder - Equivalence of games

$J_1 \equiv J_2 \iff \forall X$, $J_1 + X$ and $J_2 + X$ have the same outcome.

Refinement of $\equiv$ for subdivided bistars

Reminder - Equivalence of games

$J_1 \equiv J_2 \iff \forall X$, $J_1 + X$ and $J_2 + X$ have the same outcome.

Refinement of $\equiv$

$S \sim_1 S' \iff \forall X$, $S \overset{1}{\bullet\!-\!\bullet} X$ and $S' \overset{1}{\bullet\!-\!\bullet} X$ are equivalent.

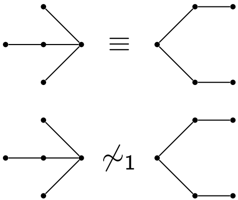
Refinement of $\equiv$ for subdivided bistars

Reminder - Equivalence of games

$J_1 \equiv J_2 \iff \forall X$, $J_1 + X$ and $J_2 + X$ have the same outcome.

Refinement of $\equiv$

$S \sim_1 S' \iff \forall X$, $S \overset{1}{\bullet\!\!-\!\!\bullet} X$ and $S' \overset{1}{\bullet\!\!-\!\!\bullet} X$ are equivalent.



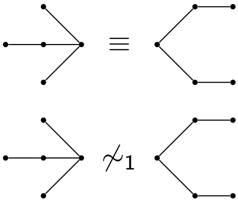
Refinement of $\equiv$ for subdivided bistars

Reminder - Equivalence of games

$J_1 \equiv J_2 \iff \forall X,\ J_1 + X$ and $J_2 + X$ have the same outcome.

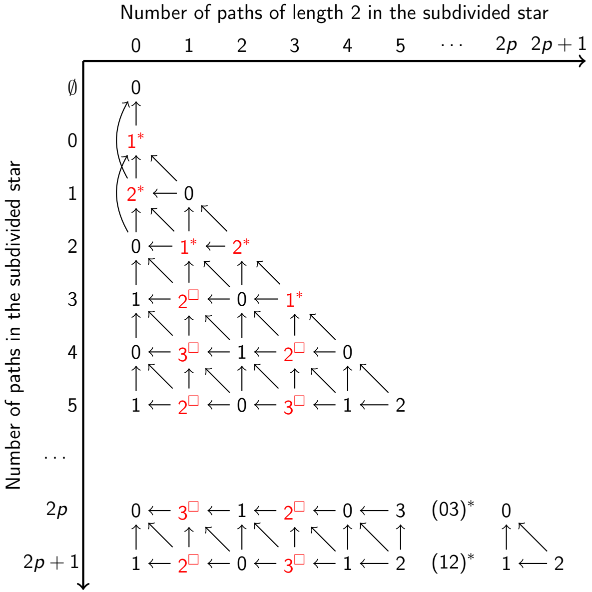
Refinement of $\equiv$

$S \sim_1 S' \iff \forall X,\ S \overset{1}{\bullet\!\!-\!\!\bullet} X$ and $S' \overset{1}{\bullet\!\!-\!\!\bullet} X$ are equivalent.



The Grundy classes will be split into several classes for $\sim_1$.

Equivalence classes of $\sim_1$ for the game 0.33



Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \overset{1}{\bullet\!-\!\bullet} S_2$ depending on the classes of S_1 and S_2 is given by:

Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \overset{1}{\bullet\!\!-\!\!\bullet} S_2$ depending on the classes of S_1 and S_2 is given by:

	0	1	1^*	2	2^*	$2^\square$	3	$3^\square$
0	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
1	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
1^*	$\oplus$	$\oplus$	2	$\oplus$	0	$\oplus$	$\oplus$	$\oplus$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
2^*	$\oplus$	$\oplus$	0	$\oplus$	1	1	$\oplus$	0
$2^\square$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	1	$\oplus$	$\oplus$	$\oplus$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
$3^\square$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	0	$\oplus$	$\oplus$	$\oplus$

where $\oplus$ is the Nim-sum.

Grundy values of subdivided bistars for the game 0.33

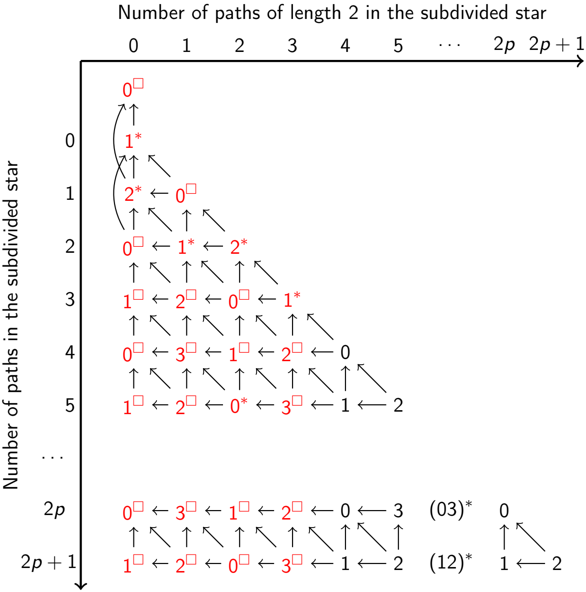
The Grundy value of $S_1 \overset{1}{\bullet\!\!-\!\!\bullet} S_2$ depending on the classes of S_1 and S_2 is given by:

	0	1	1^*	2	2^*	$2^\square$	3	$3^\square$
0	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
1	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
1^*	$\oplus$	$\oplus$	2	$\oplus$	0	$\oplus$	$\oplus$	$\oplus$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
2^*	$\oplus$	$\oplus$	0	$\oplus$	1	1	$\oplus$	0
$2^\square$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	1	$\oplus$	$\oplus$	$\oplus$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
$3^\square$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	0	$\oplus$	$\oplus$	$\oplus$

where $\oplus$ is the Nim-sum.

$\Rightarrow$ The values are still in the range $[\![0;3]\!]$

Equivalence classes of $\sim_2$ for the game 0.33



Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \overset{2}{\bullet\!\!-\!\!\bullet} S_2$ depending on the classes of S_1 and S_2 is given by:

	0	0^*	1	1^*	$1^\square$	2	2^*	$2^\square$	3	$3^\square$
0	$\oplus$	$\oplus_1$	$\oplus$	2	$\oplus_1$	$\oplus$	0	$\oplus_1$	$\oplus$	$\oplus_1$
0^*	$\oplus_1$	$\oplus_1$	$\oplus_1$	2	$\oplus_1$	$\oplus_1$	0	$\oplus_1$	$\oplus_1$	$\oplus_1$
1	$\oplus$	$\oplus_1$	$\oplus$	3	$\oplus_1$	$\oplus$	1	$\oplus_1$	$\oplus$	$\oplus_1$
1^*	2	2	3	0	3	0	1	1	1	0
$1^\square$	$\oplus_1$	$\oplus_1$	$\oplus_1$	3	$\oplus_1$	$\oplus_1$	1	$\oplus_1$	$\oplus_1$	$\oplus_1$
2	$\oplus$	$\oplus_1$	$\oplus$	0	$\oplus_1$	$\oplus$	2	$\oplus_1$	$\oplus$	$\oplus_1$
2^*	0	0	1	1	1	2	2	2	3	3
$2^\square$	$\oplus_1$	$\oplus_1$	$\oplus_1$	1	$\oplus_1$	$\oplus_1$	2	0	$\oplus_1$	1
3	$\oplus$	$\oplus_1$	$\oplus$	1	$\oplus_1$	$\oplus$	3	$\oplus_1$	$\oplus$	$\oplus_1$
$3^\square$	$\oplus_1$	$\oplus_1$	$\oplus_1$	0	$\oplus_1$	$\oplus_1$	3	1	$\oplus_1$	0

where $\oplus$ is the Nim-sum and $x \oplus_1 y$ stands for $x \oplus y \oplus 1$.

Grundy values of subdivided bistars for the game 0.33

The Grundy value of $S_1 \overset{2}{\bullet\!\!-\!\!\bullet} S_2$ depending on the classes of S_1 and S_2 is given by:

	0	0^*	1	1^*	$1^{\square}$	2	2^*	$2^{\square}$	3	$3^{\square}$
0	$\oplus$	$\oplus_1$	$\oplus$	2	$\oplus_1$	$\oplus$	0	$\oplus_1$	$\oplus$	$\oplus_1$
0^*	$\oplus_1$	$\oplus_1$	$\oplus_1$	2	$\oplus_1$	$\oplus_1$	0	$\oplus_1$	$\oplus_1$	$\oplus_1$
1	$\oplus$	$\oplus_1$	$\oplus$	3	$\oplus_1$	$\oplus$	1	$\oplus_1$	$\oplus$	$\oplus_1$
1^*	2	2	3	0	3	0	1	1	1	0
$1^{\square}$	$\oplus_1$	$\oplus_1$	$\oplus_1$	3	$\oplus_1$	$\oplus_1$	1	$\oplus_1$	$\oplus_1$	$\oplus_1$
2	$\oplus$	$\oplus_1$	$\oplus$	0	$\oplus_1$	$\oplus$	2	$\oplus_1$	$\oplus$	$\oplus_1$
2^*	0	0	1	1	1	2	2	2	3	3
$2^{\square}$	$\oplus_1$	$\oplus_1$	$\oplus_1$	1	$\oplus_1$	$\oplus_1$	2	0	$\oplus_1$	1
3	$\oplus$	$\oplus_1$	$\oplus$	1	$\oplus_1$	$\oplus$	3	$\oplus_1$	$\oplus$	$\oplus_1$
$3^{\square}$	$\oplus_1$	$\oplus_1$	$\oplus_1$	0	$\oplus_1$	$\oplus_1$	3	1	$\oplus_1$	0

where $\oplus$ is the Nim-sum and $x \oplus_1 y$ stands for $x \oplus y \oplus 1$.

$\Rightarrow$ The values are still in the range $[\![0;3]\!]$

The game 0.33 on trees

The game 0.33 on trees

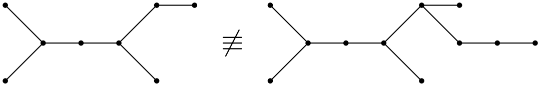
Proposition

The reduction of paths to their length modulo 3 **does not work** on trees:

The game 0.33 on trees

Proposition

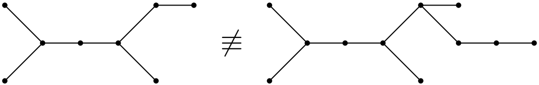
The reduction of paths to their length modulo 3 **does not work** on trees:



The game 0.33 on trees

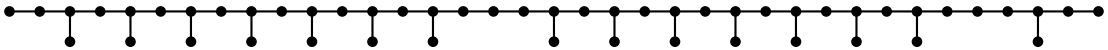
Proposition

The reduction of paths to their length modulo 3 **does not work** on trees:



Conjecture

For all $n \geq 4$, there exists a tree T such that $\mathcal{G}(T) = n$.

$\mathcal{G}($  $) = 10$

Conclusion

Summary

Conclusion

Summary

- **Natural generalization** of octal games on graphs;

Conclusion

Summary

- **Natural generalization** of octal games on graphs;
- **Complete resolution** of 0.33 on subdivided stars and bistars: every path can be **reduced** to its **length modulo 3**;

Conclusion

Summary

- **Natural generalization** of octal games on graphs;
- **Complete resolution** of 0.33 on subdivided stars and bistars: every path can be **reduced** to its **length modulo 3**;
- Expression of the Grundy value of a subdivided bistar as a **pseudo-sum** of its two stars' Grundy values;

Conclusion

Summary

- **Natural generalization** of octal games on graphs;
- **Complete resolution** of 0.33 on subdivided stars and bistars: every path can be **reduced** to its **length modulo 3**;
- Expression of the Grundy value of a subdivided bistar as a **pseudo-sum** of its two stars' Grundy values;
- The result **does not hold** for trees.

Conclusion

Summary

- **Natural generalization** of octal games on graphs;
- **Complete resolution** of 0.33 on subdivided stars and bistars: every path can be **reduced** to its **length modulo 3**;
- Expression of the Grundy value of a subdivided bistar as a **pseudo-sum** of its two stars' Grundy values;
- The result **does not hold** for trees.

Perspectives

- Prove that trees can have **arbitrarily large** Grundy values;

Conclusion

Summary

- **Natural generalization** of octal games on graphs;
- **Complete resolution** of 0.33 on subdivided stars and bistars: every path can be **reduced** to its **length modulo 3**;
- Expression of the Grundy value of a subdivided bistar as a **pseudo-sum** of its two stars' Grundy values;
- The result **does not hold** for trees.

Perspectives

- Prove that trees can have **arbitrarily large** Grundy values;
- Studying other graph classes;

Conclusion

Summary

- **Natural generalization** of octal games on graphs;
- **Complete resolution** of 0.33 on subdivided stars and bistars: every path can be **reduced** to its **length modulo 3**;
- Expression of the Grundy value of a subdivided bistar as a **pseudo-sum** of its two stars' Grundy values;
- The result **does not hold** for trees.

Perspectives

- Prove that trees can have **arbitrarily large** Grundy values;
- Studying other graph classes;
- **Generalize** some results on other octal games.

Conclusion

Summary

- **Natural generalization** of octal games on graphs;
- **Complete resolution** of 0.33 on subdivided stars and bistars: every path can be **reduced** to its **length modulo 3**;
- Expression of the Grundy value of a subdivided bistar as a **pseudo-sum** of its two stars' Grundy values;
- The result **does not hold** for trees.

Perspectives

- Prove that trees can have **arbitrarily large** Grundy values;
- Studying other graph classes;
- **Generalize** some results on other octal games.