Connected subtraction games on graphs

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Joint work with Julien Moncel (LAAS) and Aline Parreau (LIRIS).

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CGTC3, January 22nd 2019

Subtraction Games

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- Played on heaps of counters
- Removing k counters from a heap $\Leftrightarrow k \in S$

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Studying subtraction games

Sprague-Grundy Theorem $\Rightarrow$ One heap is sufficient

Periodicity results

Grundy sequence

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If S is finite, then the Grundy sequence of SUB(S) is ultimately periodic.

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Theorem (Albert, Nowakowski, Wolfe, 2007)

If S is finite, then the Grundy sequence of $\mathrm{SUB}(\mathbb{N} \setminus S)$ is ultimately arithmetic periodic.

Going further

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We will play subtraction games on graphs !

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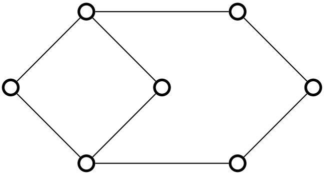

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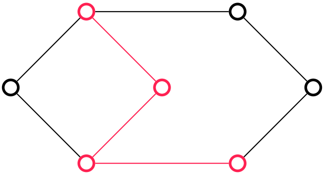

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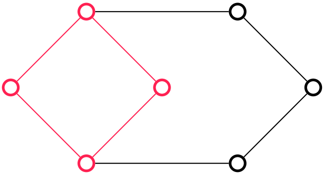

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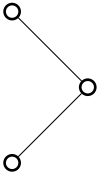

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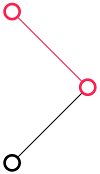

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- On heaps $\rightarrow$ Grundy sequence
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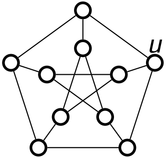
Idea

Studying the Grundy values when appending a path to a given vertex.

$G \underset{u}{\bullet\!\!-\!\!\bullet} P_k =$  $(k = 0)$

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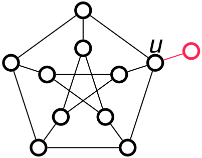
$G \underset{u}{\bullet\!\!-\!\!\bullet} P_k =$ 

$(k = 1)$

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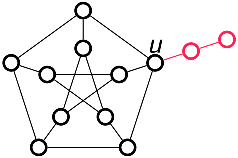
$G \underset{u}{\bullet\!\!-\!\!\bullet} P_k =$ 

$(k = 2)$

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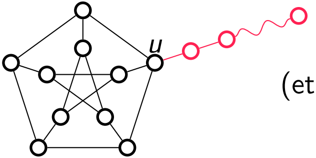

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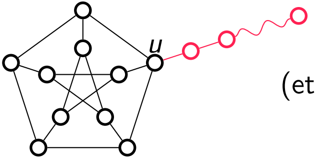

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$\rightarrow$ Used for NODE-KAYLES (Fleischer and Trippen, 2004) and ARC-KAYLES (Huggan and Stevens, 2016)

What we already know

Not subtraction games

- NODE-KAYLES : Ultimate periodicity for P_k with u as the second vertex (*i.e.* $S_{1,k-2,\ell}$)
- ARC-KAYLES : Ultimate periodicity test for subdivided stars with three paths with u as a leaf

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Subtraction games

- S finite : Ultimate periodicity for paths with u as a leaf
- CSG($\{1, 2\}$) :Periodicity for subdivided stars and bistars with u as the central vertex or a leaf

A general periodicity result

Theorem (D., Moncel, Parreau, 2019+)

If S is finite, then for every graph G and vertex u the sequence of the $G \underset{u}{\bullet\!\!-\!\!\bullet} P_k$ for CSG(S) is ultimately periodic.

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Idea of the proof

Induction on $|G|$.

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2. Otherwise, three possible kinds of moves :
   2.1 Playing on $P_k \rightarrow |S|$ different moves
   2.2 Playing on G without removing $u \rightarrow$ at most $2^{|G|-1}$ different moves
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   $\Rightarrow \mathcal{G}(G \underset{u}{\bullet}\!\!-\!\!\bullet P_k) \leq C$

   Every move brings us to a periodic sequence, the result comes from mex computation.

Questions

1. What period for a connected subtraction game ?
2. Which games are purely periodic, and on which graphs ?
3. What about CSG($\mathbb{N} \setminus S$) ?

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2. Which games are purely periodic, and on which graphs ?
3. What about $\mathrm{CSG}(\mathbb{N} \setminus S)$?

The family $\mathrm{CSG}(\{1, \ldots, N\})$

Removing $1, 2, \ldots, N$ connected vertices, without disconnecting the graph.

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Question

Are there graph families such that the sequence of the $G \underset{u}{\bullet\!\!-\!\!\bullet} P_k$ has period $N + 1$?

CSG($\{1, \ldots, N\}$) on stars

Theorem (D., Moncel, Parreau, 2019+)

If G is a star, then the sequence of the $G \underset{u}{\bullet\!\!-\!\!\bullet} P_k$ is periodic with period $N + 1$.

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If $G = S_{1,\ell}$ and u is the central vertex, then the sequence of the $G \underset{u}{\bullet\!\!-\!\!\bullet} P_k$ is ultimately periodic with period $N+1$.

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Remark

Purely periodic for some values of ℓ, preperiod $N + 1$ for others.

And subdivided stars?

Periodicity for $\text{CSG}(\{1,2\})$ and $\text{CSG}(\{1,2,3\})$ if u is the central vertex or a leaf, with period $N+1$.

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Drawback

Many base cases...

- For CSG($\{1,2\}$) : 2 base cases
- For CSG($\{1,2,3\}$) : 7 base cases

Adjoining integers to a subtraction set

Definition

If, for all n, $\mathcal{G}(n)$ is the same for $\mathrm{SUB}(S)$ and $\mathrm{SUB}(S \cup \{k\})$, then k can be adjoined to S.

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What about graphs ?

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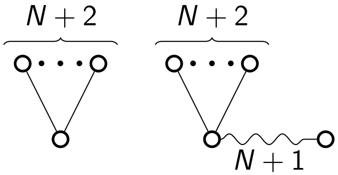

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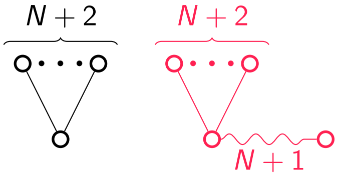

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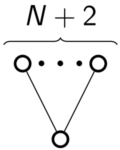

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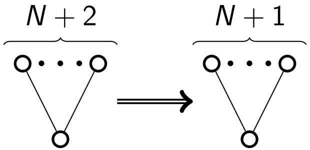

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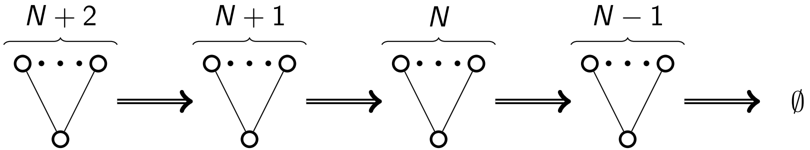

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