A Canadian is traveling on an outerplanar graph...

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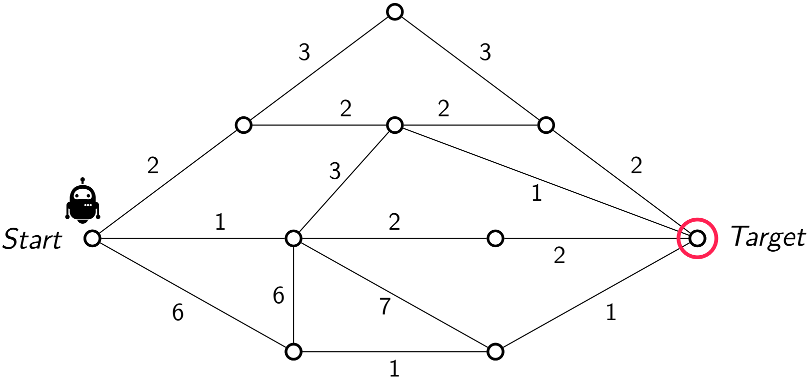
The Canadian Traveler



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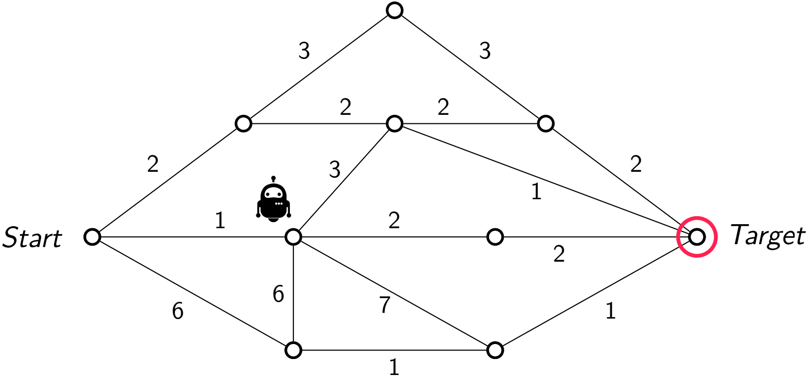

Traveled distance $= 1$

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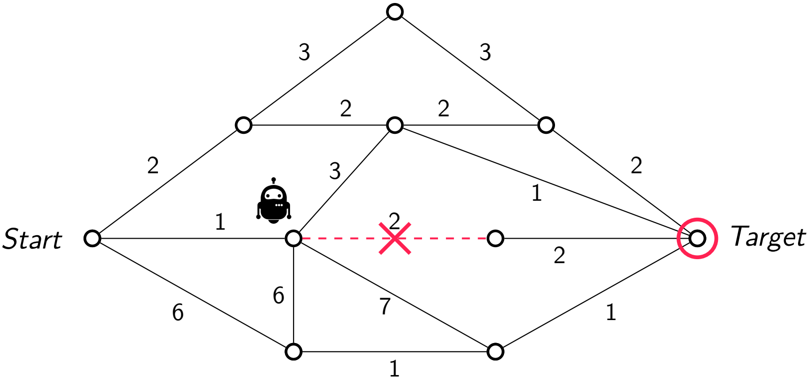

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- Some edges are **blocked**, discovered when visiting an endpoint

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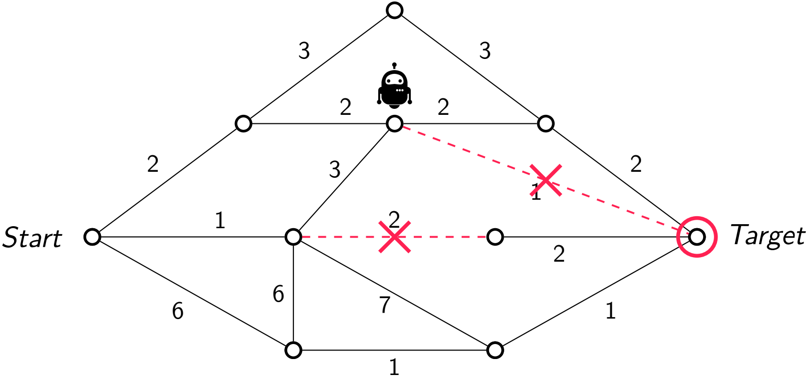

Traveled distance = 4

- Some edges are **blocked**, discovered when visiting an endpoint

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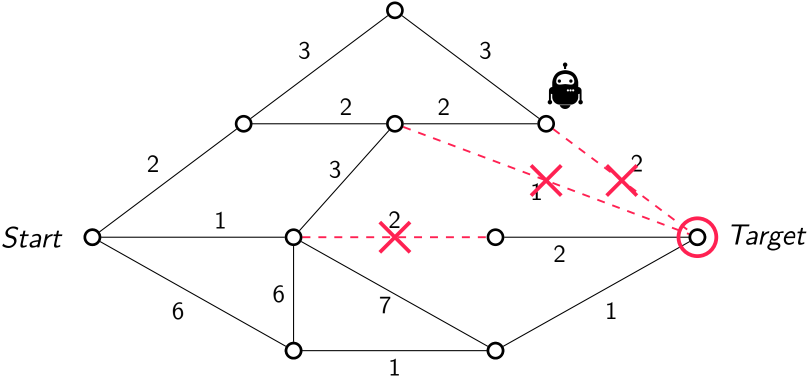

Traveled distance = 6

- Some edges are **blocked**, discovered when visiting an endpoint

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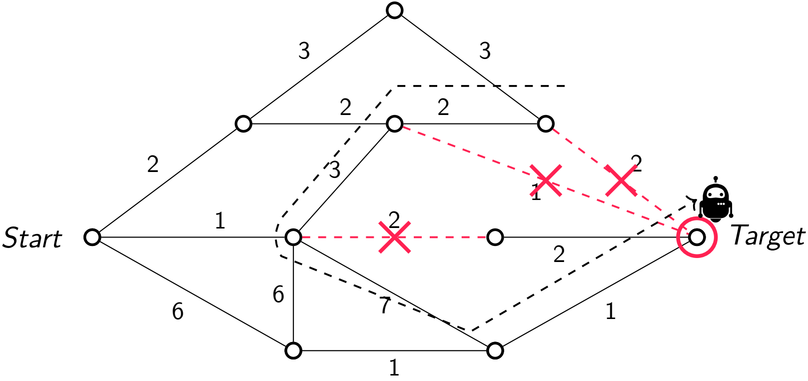

Traveled distance $= 19$

- Some edges are **blocked**, discovered when visiting an endpoint
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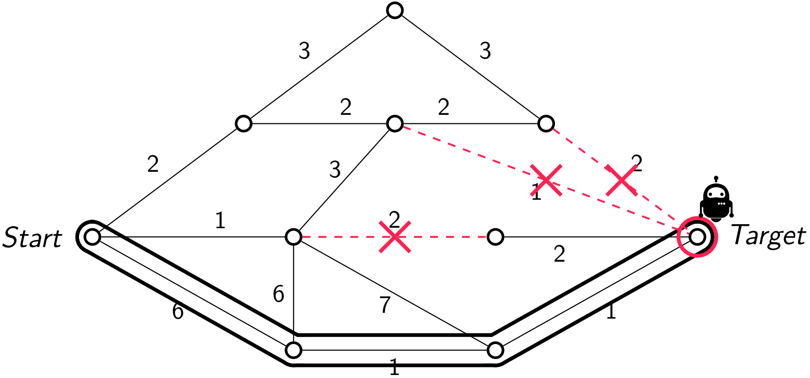

Traveled distance $= 19$
Optimal distance $= 8$

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Formal definition and bad news

k-CTP [Papadimitriou & Yannakakis, 1991]

Input: A weighted graph, two vertices s and t.
Hidden input: At most k blocked edges.
Objective: Go from s to t with minimum traveled distance.

Evaluating a strategy

Minimizing the **competitive ratio** $\frac{\text{traveled distance}}{\text{optimal distance}}$

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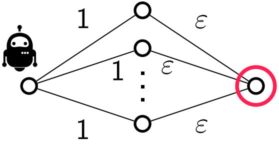

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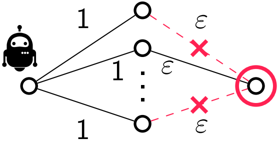

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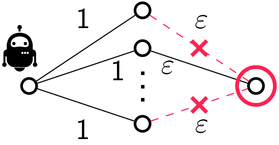

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The construction can even be made unit-weighted...

A bit of history

- k-CTP is PSPACE-complete [Papadimitriou & Yannakakis and Bar-Noy & Schieber, 1991]
- Many variants (probabilistic, multiple travelers, sensing remote edges, temporal graphs...) with applications to robot routing
- The Greedy strategy (follow a shortest path from s to t, when blocked at x, compute a shortest path from x to t) can be arbitrarily bad
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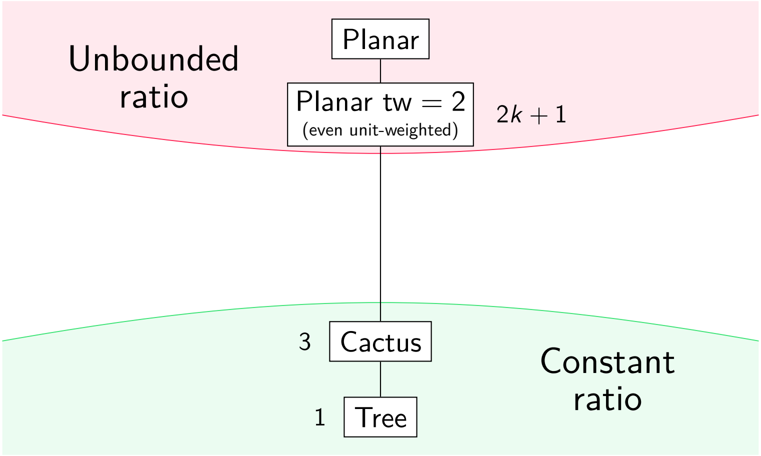
  - COMPARISON [Xu, Hu, Su, Zhu & Zhu, 2009]: trade-off between GREEDY and REPOSITION
- Non-deterministic strategies can be quite good: competitive ratio $(1 + \frac{\sqrt{2}}{2})k + O(1)$ [Demaine *et al.*, 2014], but no better than $k + 1$ [Westphal, 2008]

What now?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Planar
Planar tw = 2
(even unit-weighted)
2k + 1
3
Cactus
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Tree
Constant ratio

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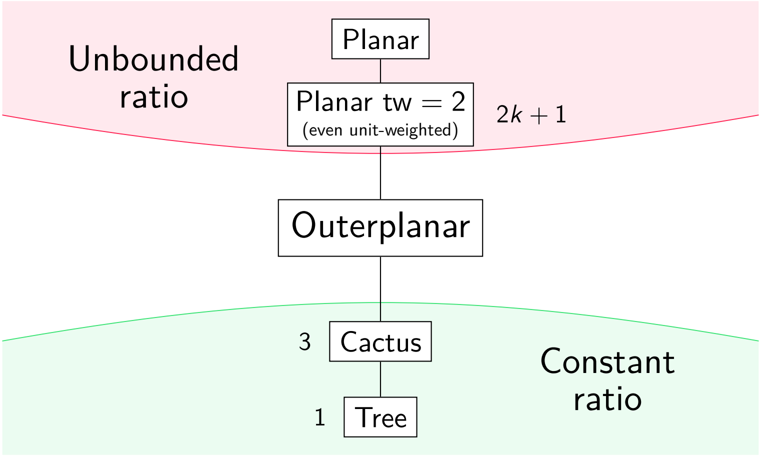

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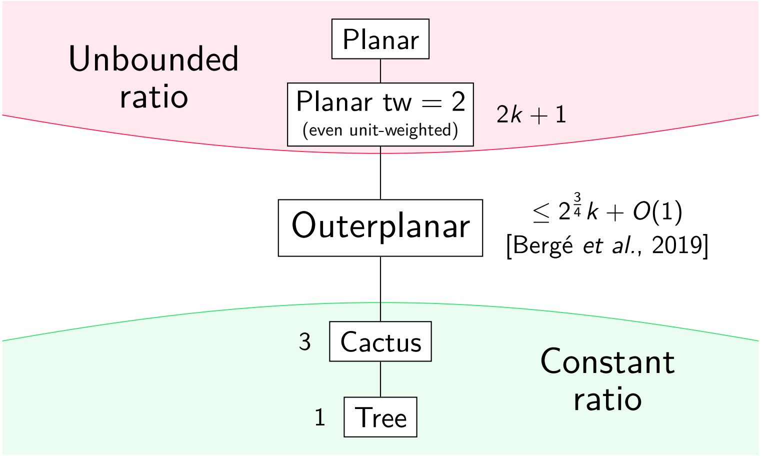
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[Bergé *et al.*, 2019]
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Our results

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There is a strategy with competitive ratio 9 for unit-weighted outerplanar graphs.

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There is a family of unit-weighted outerplanar graphs on which no strategy can have competitive ratio < 9.

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Shells and cows

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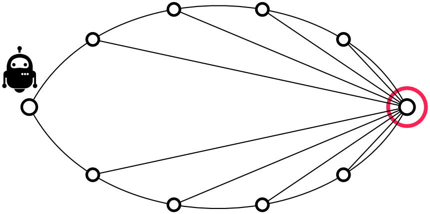

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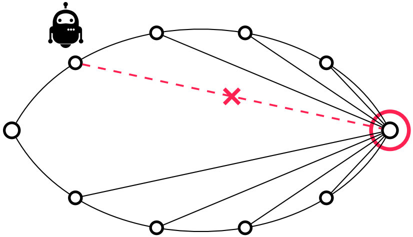

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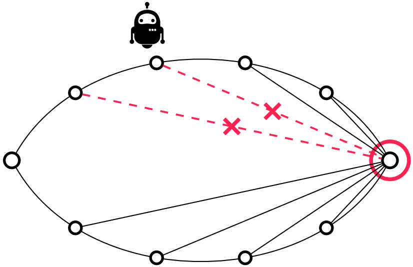

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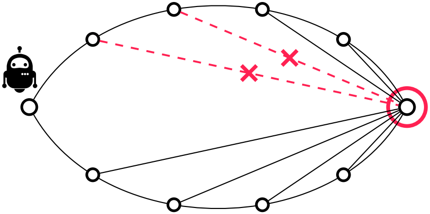

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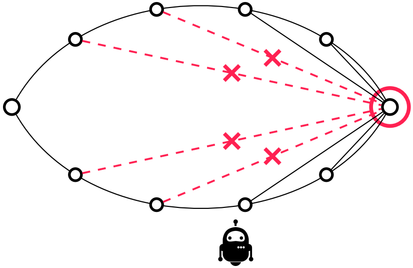

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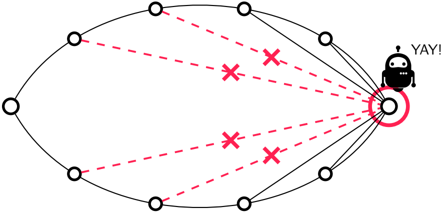

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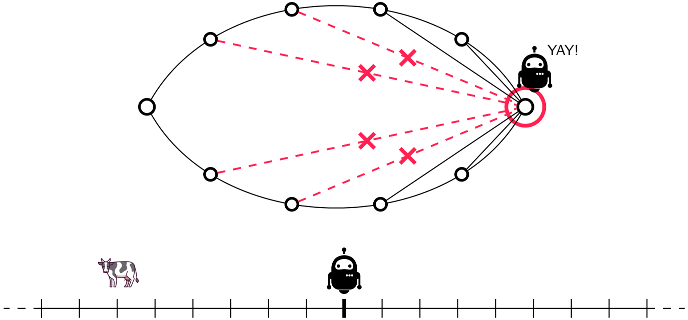

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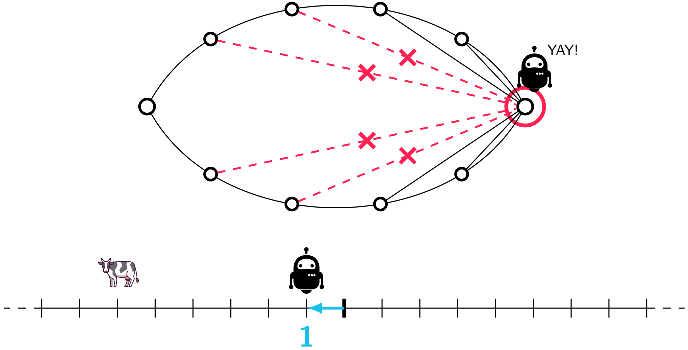

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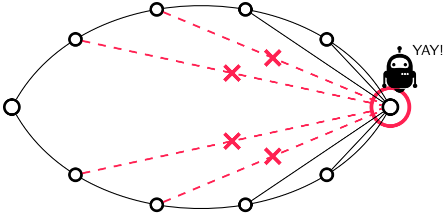

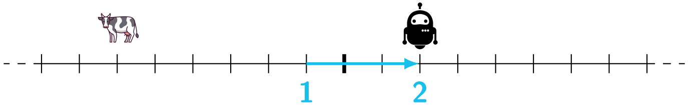

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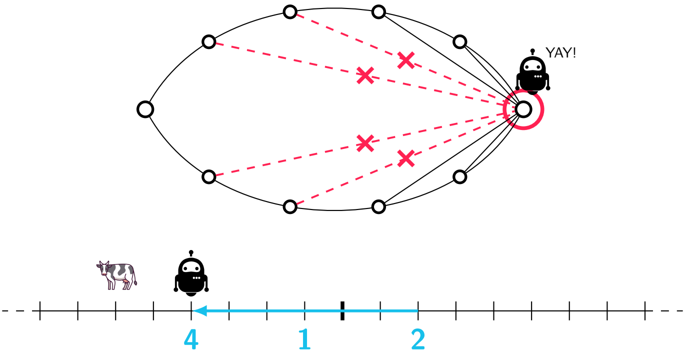

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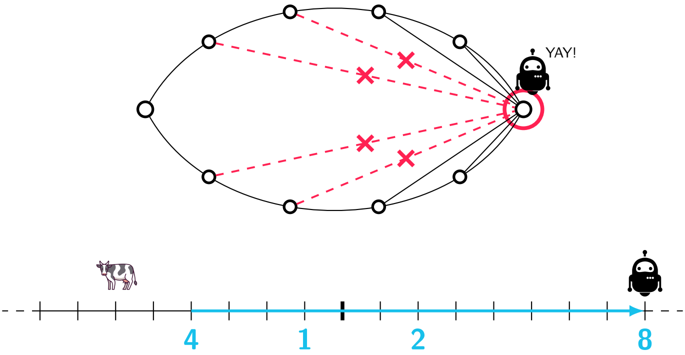

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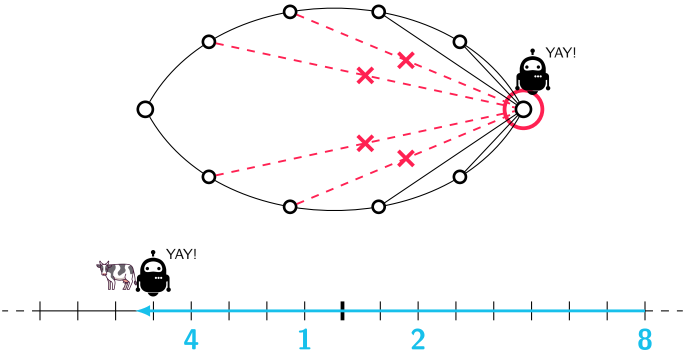

$\rightarrow$ No strategy can be better than this, which has ratio 9

Competitive ratio 9 on unweighted outerplanar graphs

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Proof Sketch

1. Manage articulation points to simplify and decompose the graph
2. Use exponential balancing while managing chords, using induction to iterate

Main idea: going from one side to the other without going back to the start can be useful!

Managing articulation points

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Lemma

Let $\mathcal{F}$ be a monotone family. If A achieves competitive ratio C on graphs of $\mathcal{F}$ with no articulation point, then, the strategy:

1. Remove all useless components
2. For every (s, t)-separator z, apply A from s to z then from z to t

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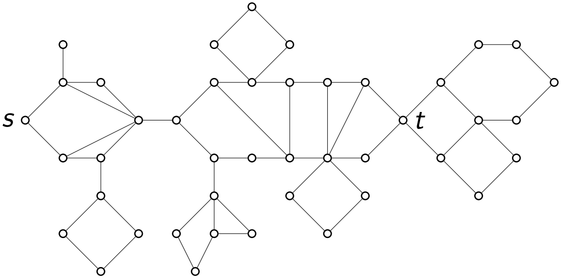

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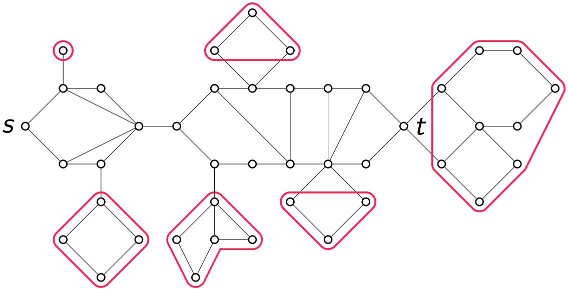

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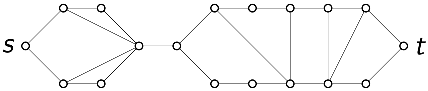

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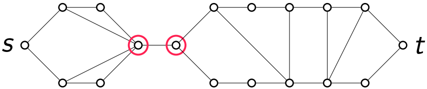

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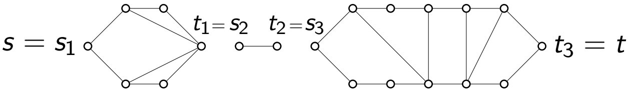
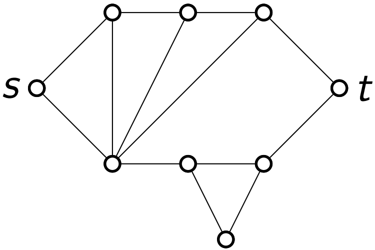


Sides and chords



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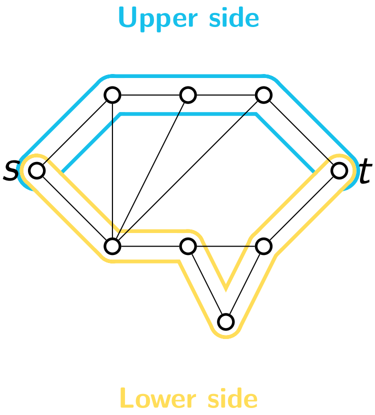

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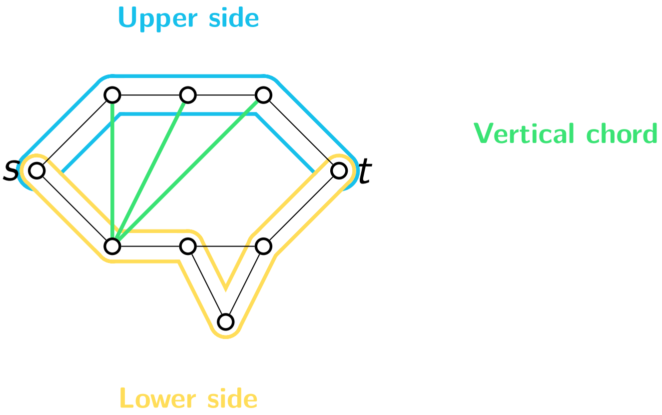

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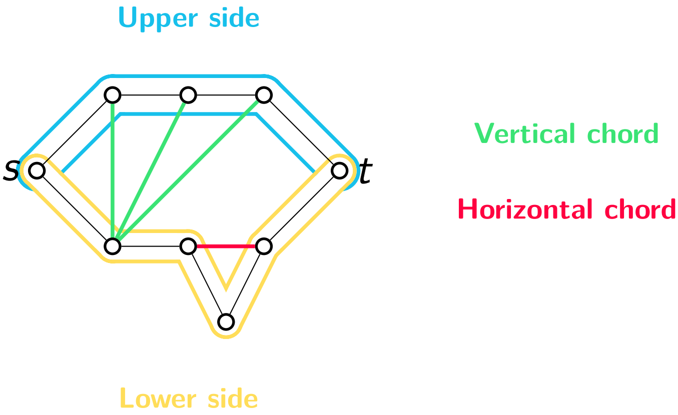

→ When following a path, we always take open horizontal chords and can ignore what they allow to skip

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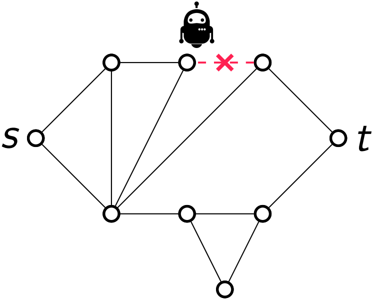

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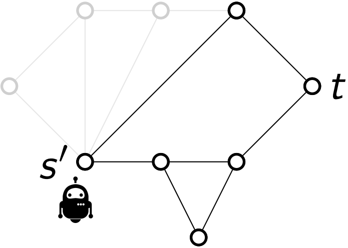
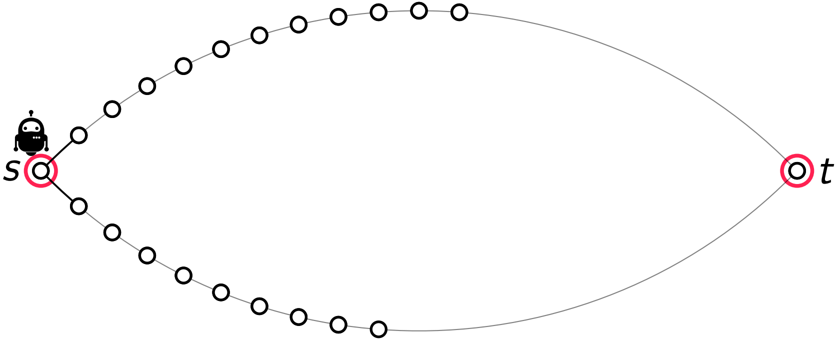


$\rightarrow$ When following a path, we always take open horizontal chords and can ignore what they allow to skip
$\rightarrow$ If one side is blocked, then, we switch to the other and can reapply the simplification

Exponential balancing and vertical chords



Exponential balancing and vertical chords

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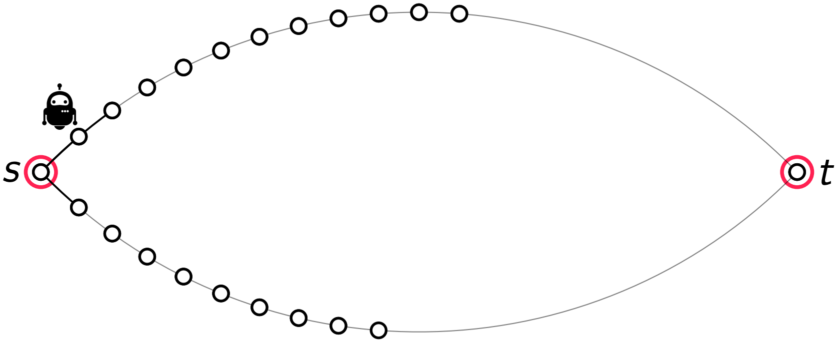

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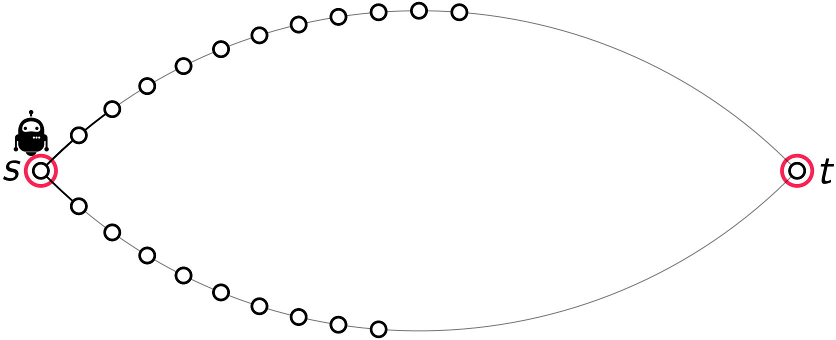

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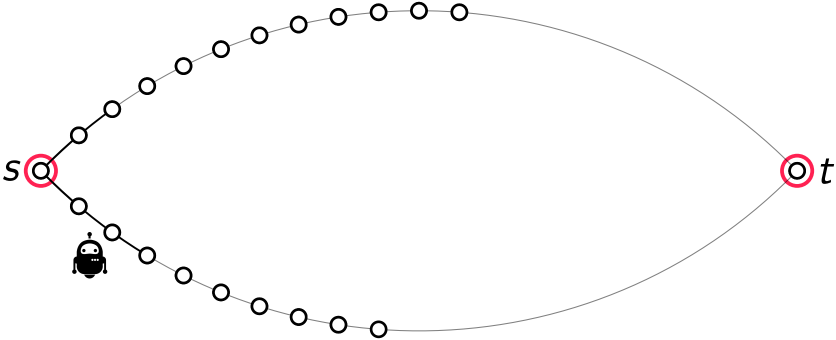

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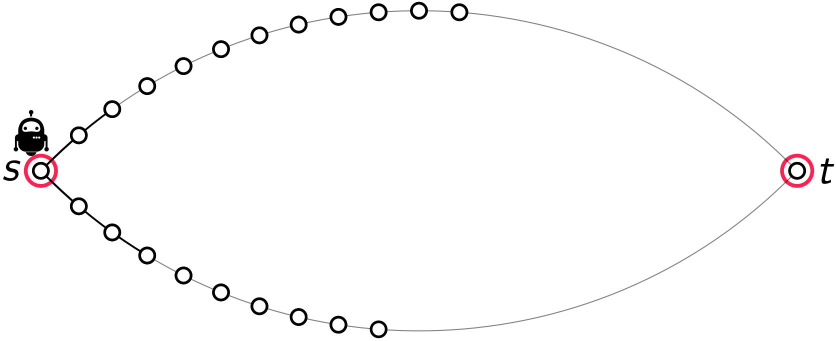

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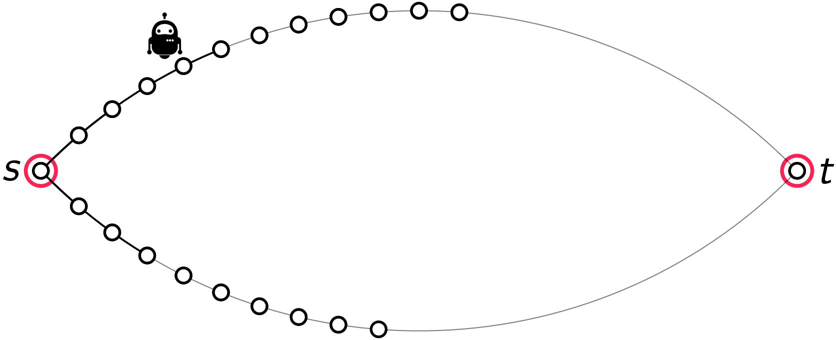
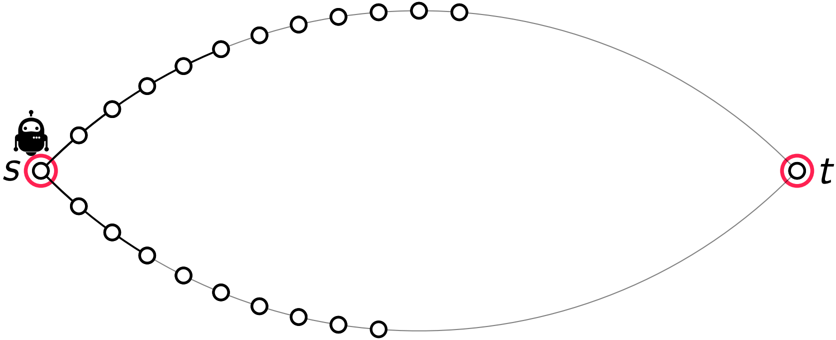


Exponential balancing and vertical chords

Budget $= 8$



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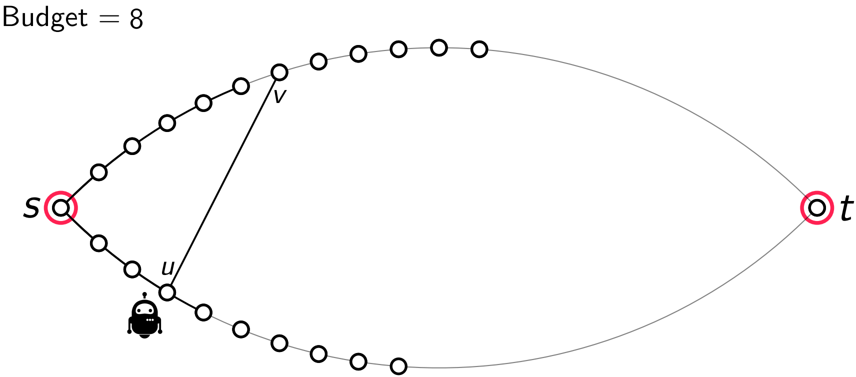

Case 1: When catching up, a vertical chord allows us to go further on the other side than previous budget.

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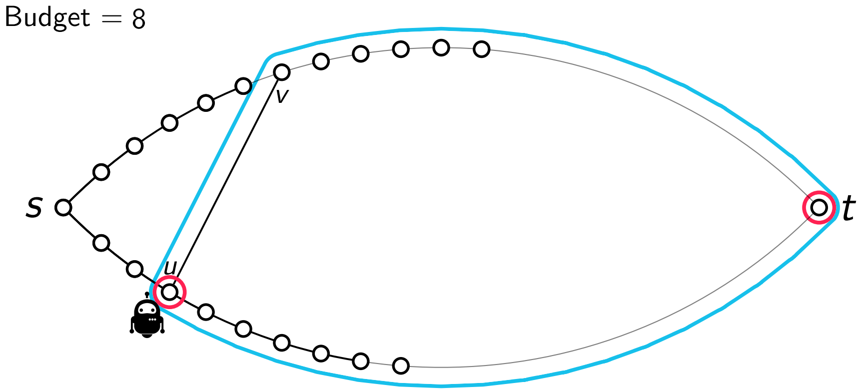

Case 1: When catching up, a vertical chord allows us to go further on the other side than previous budget.

$\Rightarrow$ Shortest *sv*-path goes through *u*

$\Rightarrow$ **Iterate** from *u* to *t*

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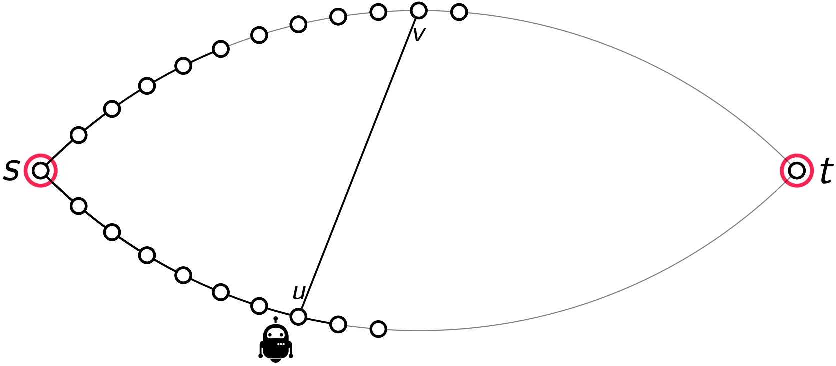
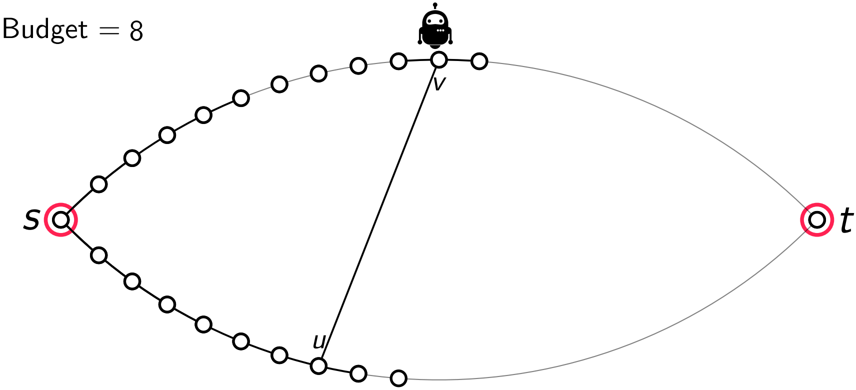


Case 2: When exploring further than previous budget, a vertical chord links us to an unexplored vertex on the other side.

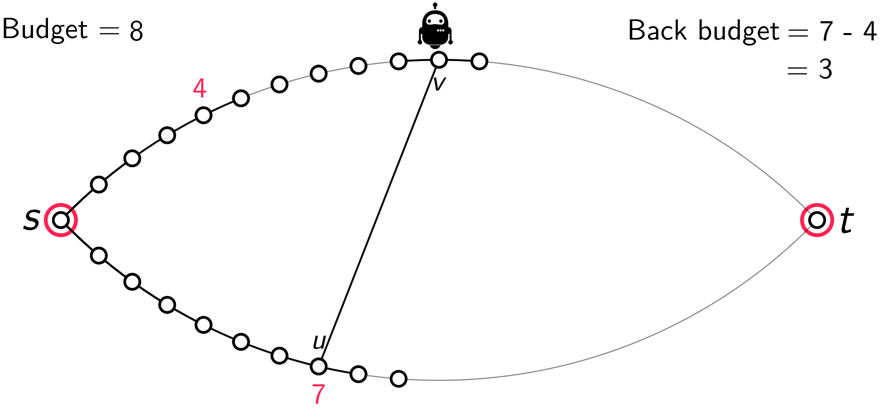
Exponential balancing and vertical chords



Case 2: When exploring further than previous budget, a vertical chord links us to an unexplored vertex on the other side.

$\Rightarrow$ Go to the other side and explore **back**

Exponential balancing and vertical chords



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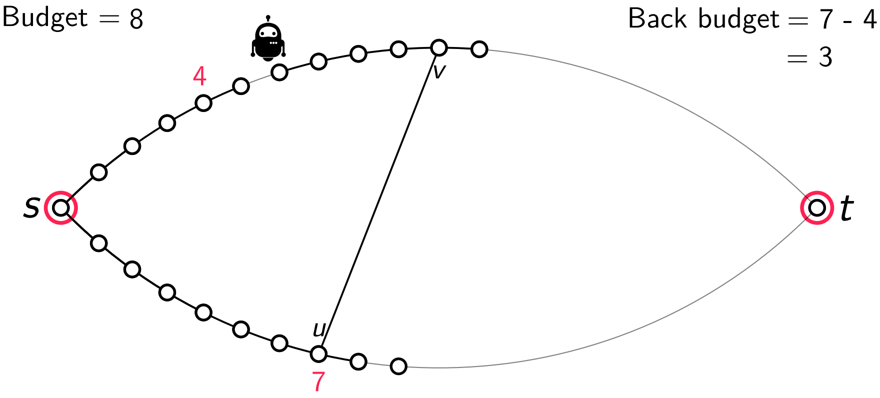
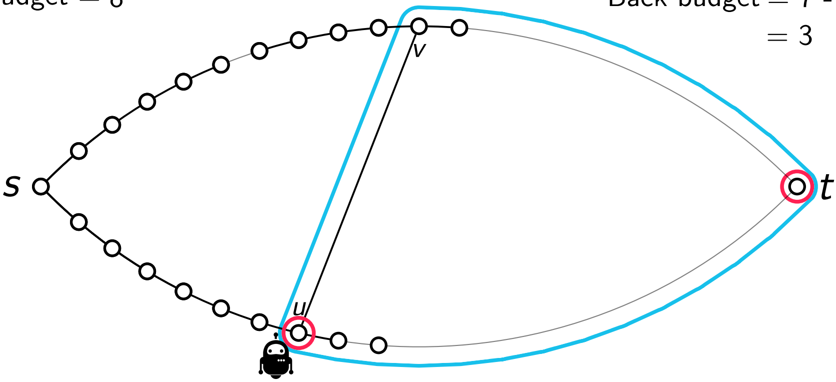


Case 2.1: We do not reach the last explored vertex on the other side.

Exponential balancing and vertical chords

Budget $= 8$

Back budget $=$ 7 - 4
$= 3$

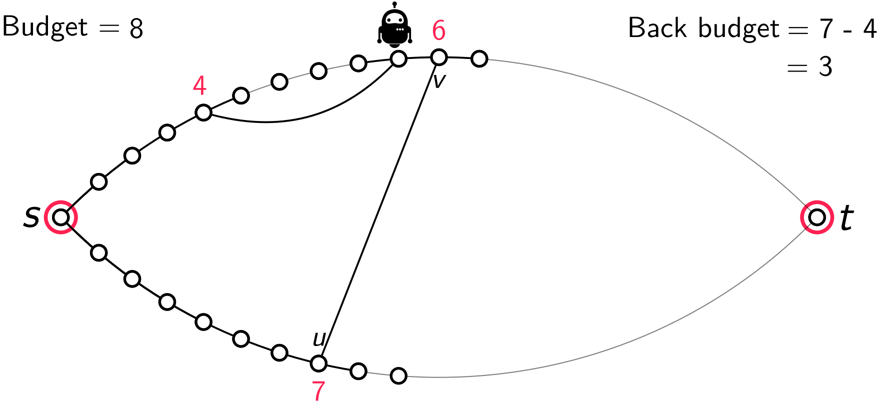


Case 2.1: We do not reach the last explored vertex on the other side.

$\Rightarrow$ Shortest sv-path goes through u

$\Rightarrow$ **Iterate** from u to t

Exponential balancing and vertical chords



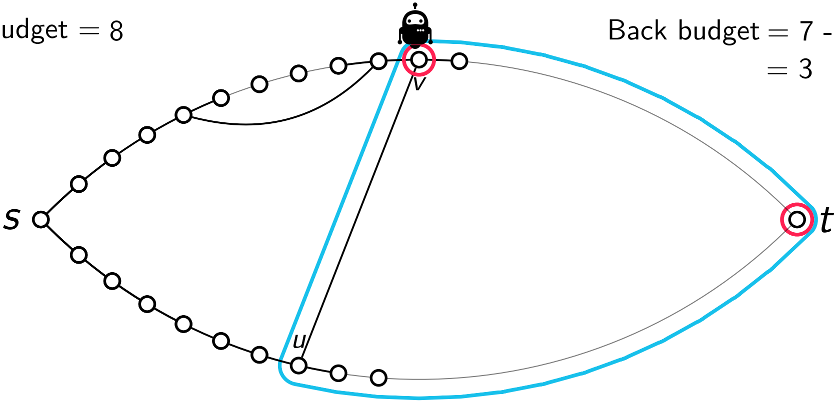
Case 2.2: We reach the last explored vertex on the other side.

Exponential balancing and vertical chords

Budget = 8

Back budget = 7 - 4

= 3

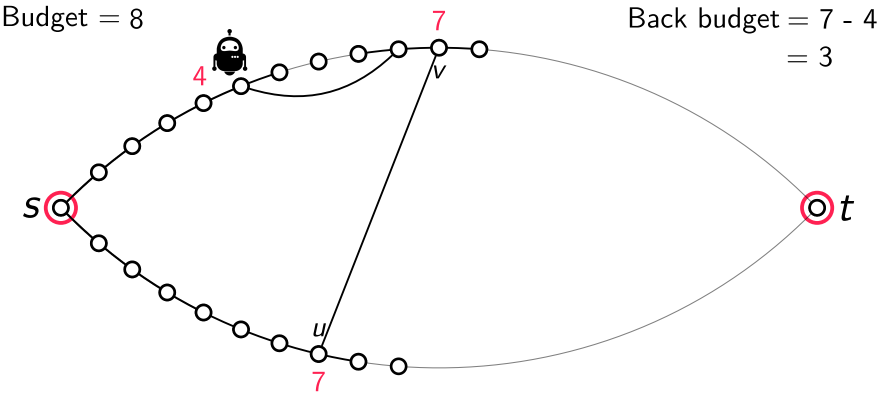


Case 2.2: We reach the last explored vertex on the other side.

$\Rightarrow$ Shortest *su*-path goes through v

$\Rightarrow$ **Iterate** from v to t

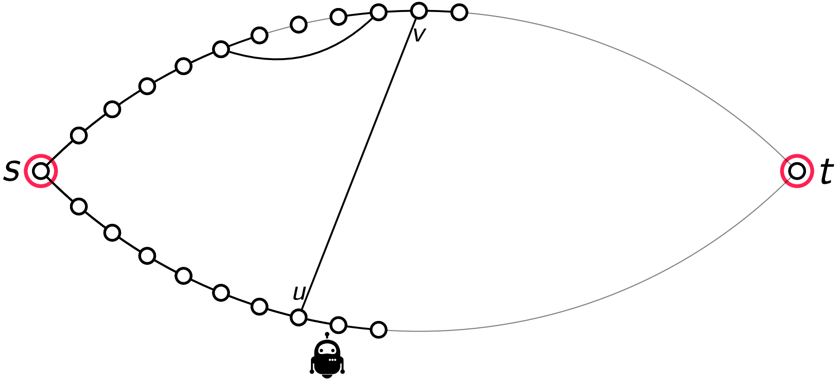
Exponential balancing and vertical chords



Case 2.3: u and v are at the same distance from s on their sides.

Exponential balancing and vertical chords

Budget $= 8$

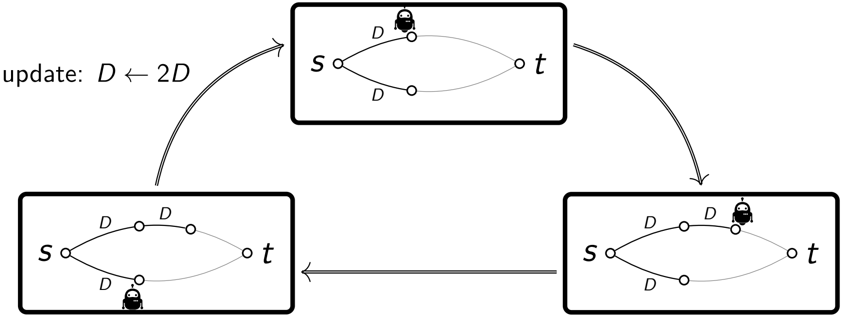


Case 2.3: u and v are at the same distance from s on their sides.

$\Rightarrow$ uv is just a shortcut between sides

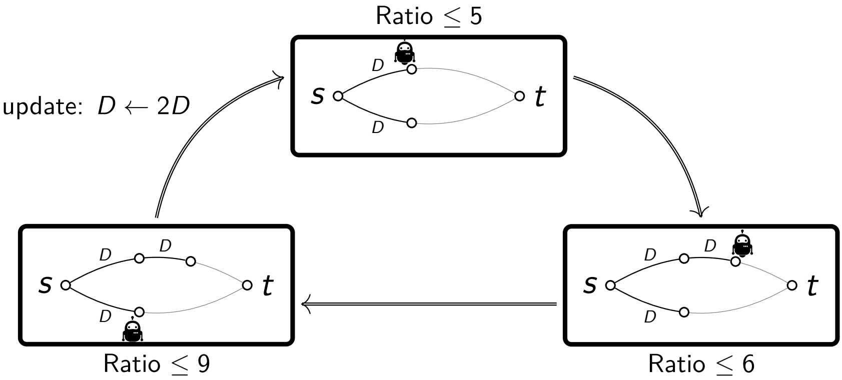
$\Rightarrow$ **Continue the balancing** from s to t

Proof of the ratio



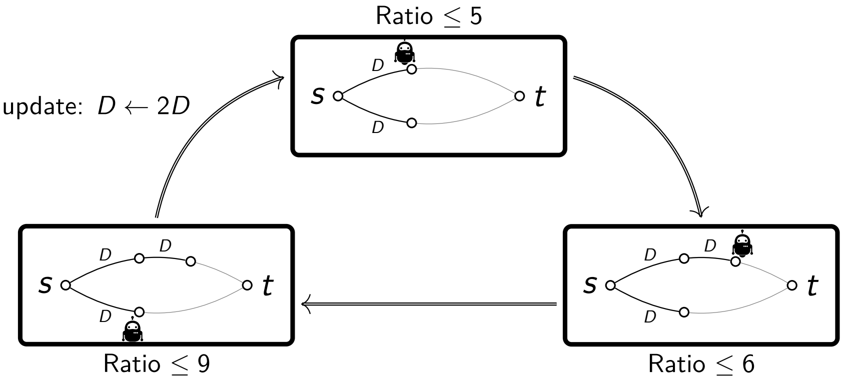
- Core **exponential balancing** loop:

Proof of the ratio



- Core **exponential balancing** loop: ratio ≤ 9

Proof of the ratio



- Core **exponential balancing** loop: ratio ≤ 9
- Going back within leftover budget **ensures** that the ratio remains ≤ 9

Arbitrarily weighted outerplanar graphs

Theorem [BBCDGLLP, 2024]

There is a family of weighted outerplanar graphs on which no strategy can have constant competitive ratio.

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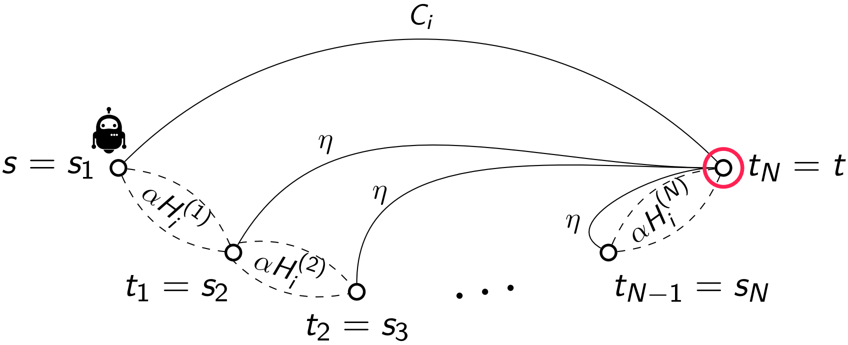
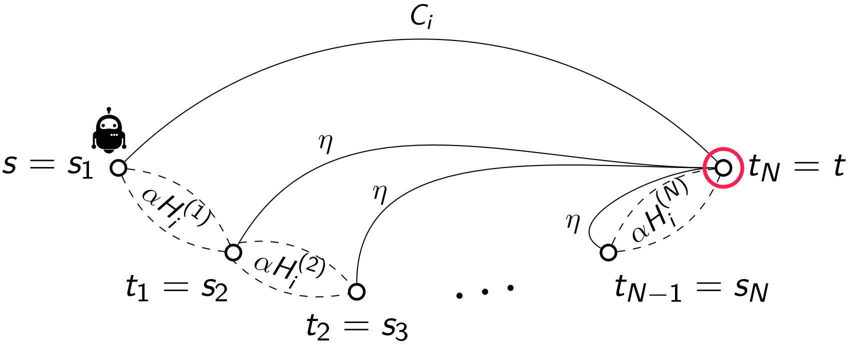
Competitive ratio $1 > \frac{1}{2}$

Building H_{i+1} from H_i

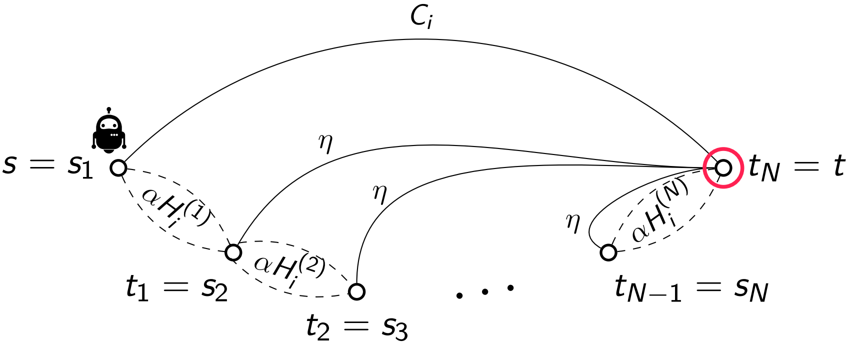


Building H_{i+1} from H_i



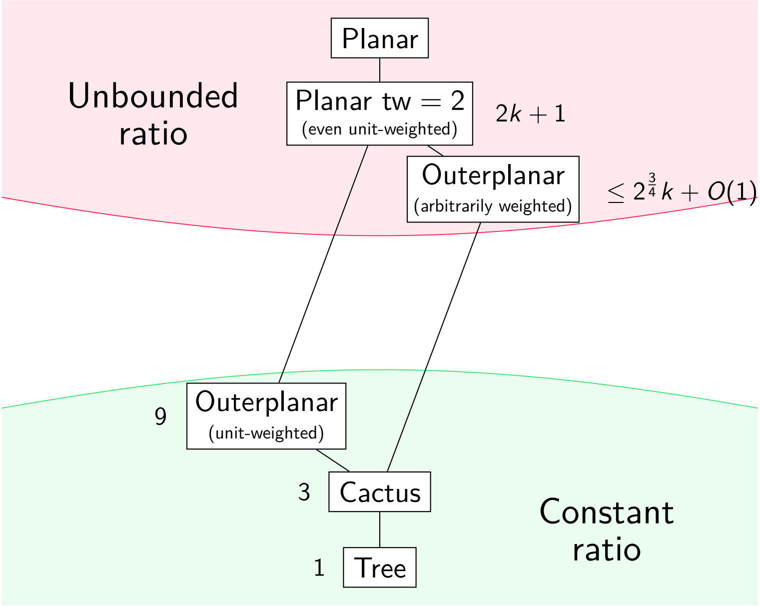
- Strategy: either crossing st, or going down. If going down, either ending up at t, or going back to s to cross st.

Building H_{i+1} from H_i



- Strategy: either crossing st, or going down. If going down, either ending up at t, or going back to s to cross st.
- Carefully choosing α and η small enough and N large enough gives competitive ratio $> C_i + 1 = C_{i+1}$.

Summary



Planar
Unbounded ratio
Planar tw = 2
(even unit-weighted)
$2k+1$
Outerplanar
(arbitrarily weighted)
$\leq 2^{\frac{3}{4}}k + O(1)$
9
Outerplanar
(unit-weighted)
3
Cactus
Constant ratio
1
Tree

Summary and perspectives!