Balanceability

Antoine Dailly

Joint work with Laura Eslava[1], Adriana Hansberg[2],
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[2] Instituto de Matemáticas, UNAM Juriquilla

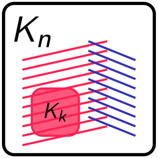
Context: Ramsey Theory

Idea

Guarantee the existence of ordered substructures within large chaotic structures.

Ramsey's Theorem (for 2-colorings) (1930)

For any k, there is an integer $R(k)$ such that, if $n \geq R(k)$, then, every 2-edge-coloring of K_n contains a monochromatic K_k.



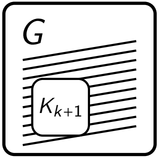
Context: extremal graph theory

Idea

Find the minimum density guaranteeing a given property, and the densest graphs for which it does not hold.

Turán's Theorem (1941)

If G of order n contains more than $\left(1-\frac{1}{k}\right)\frac{n^2}{2}$ edges, then, G contains a K_{k+1}.
The extremal graph is the balanced complete k-partite graph of order n.



Notations for the rest of the talk

- We consider 2-edge-colorings of K_n: $E(K_n) = R \sqcup B$.
- We denote by $\mathrm{ex}(n, G)$ the maximum number of edges in a G-free graph of order n.

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Goal: generalizing Ramsey's ideas and looking for unavoidable patterns other than monochromatic copies.

r-tonality

Definition

An (r, b)-copy of a graph $G(V, E)$ (with $r + b = |E|$) is a copy of G with r edges in R and b edges in B.

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r-tonality

If, for every n large enough, there exists $k(n,r)$ such that every 2-edge-coloring $R \sqcup B$ of K_n verifying $|R|,|B|>k(n,r)$ contains an (r,b)-copy of G, then G is r-tonal.

Balanceability: when $r = \frac{|E|}{2}$

Balanced copy

A balanced copy of $G(V, E)$ is an (r, b)-copy of G with $r \in \left\{ \left\lfloor \frac{|E|}{2} \right\rfloor, \left\lceil \frac{|E|}{2} \right\rceil \right\}$.

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Balanceability (Caro, Hansberg, Montejano, 2020)

Let $\mathrm{bal}(n,G)$ be the smallest integer, if it exists, such that every 2-edge-coloring $R \sqcup B$ of K_n verifying $|R|,|B| > \mathrm{bal}(n,G)$ contains a balanced copy of G.

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If there is an n_0 such that, for every $n \geq n_0$, $\mathrm{bal}(n,G)$ exists, then G is balanceable and $\mathrm{bal}(n,G)$ is its balancing number.

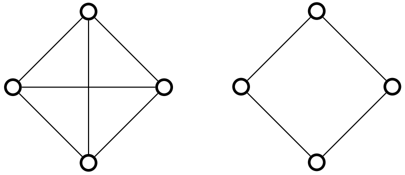
Ramsey-type problem

Extremal-type problem

Characterization

Theorem (Caro, Hansberg, Montejano, 2020)

A graph is balanceable if and only if it has both:



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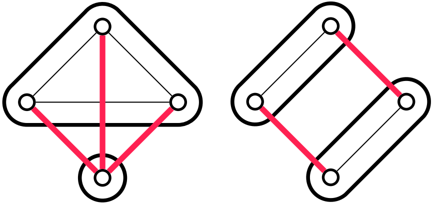

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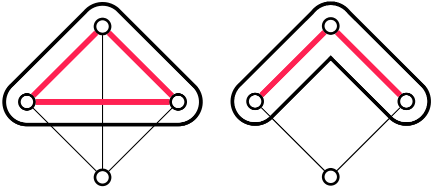
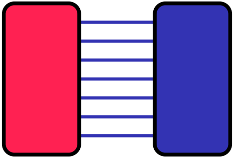
2. An induced subgraph containing half of its edges.

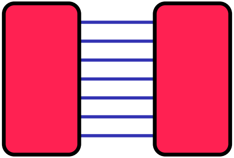


Proof of the characterization (1)

G is balanceable $\Rightarrow$
It has to fit in those two specific colorings of K_n :



type A



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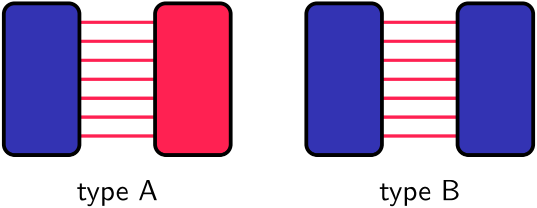

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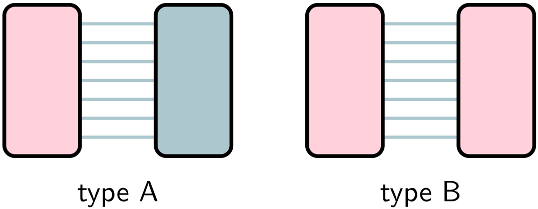

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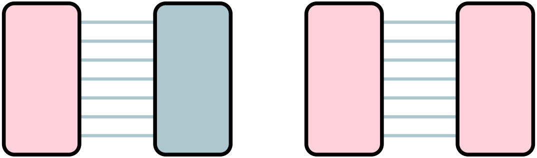


type A type B

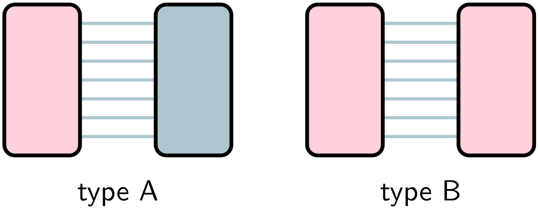
Those two specific colorings of K_n can be balanced ($|R| = |B| = \frac{1}{2}\binom{n}{2}$) for an infinity of values of n.

Proof of the characterization (2)

Theorem (Caro, Hansberg, Montejano, 2020)

For every t, there exists $\phi(n,t) \in \mathcal{O}(n^{2-\frac{1}{m(t)}})$ such that, if n is large enough, then, every 2-edge-coloring of K_n verifying $|R|, |B| \geq \phi(n,t)$ contains either a type A or a type B copy of K_{2t}.

Also shown (with a bound of $\epsilon\binom{n}{2}$) by Cutler & Montágh (2008) and Fox & Sudakov (2008).



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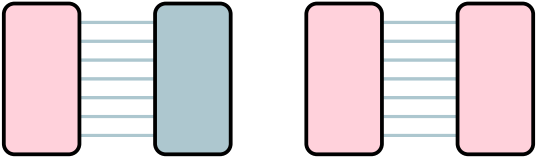

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$\Rightarrow$ Gives a subquadratic bound for $\mathrm{bal}(n, G)$

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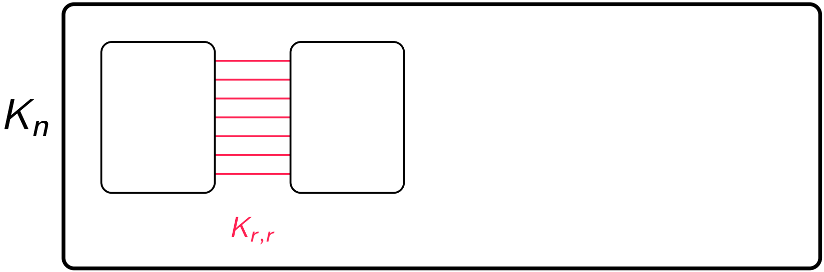

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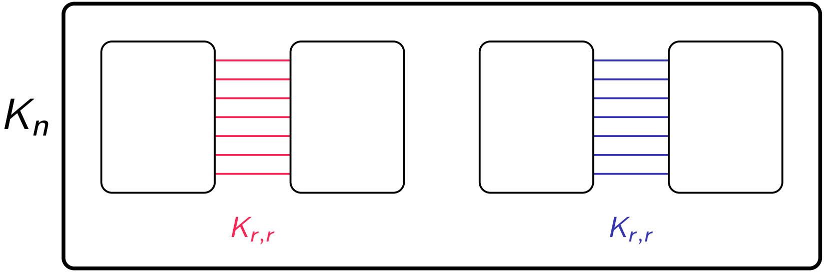

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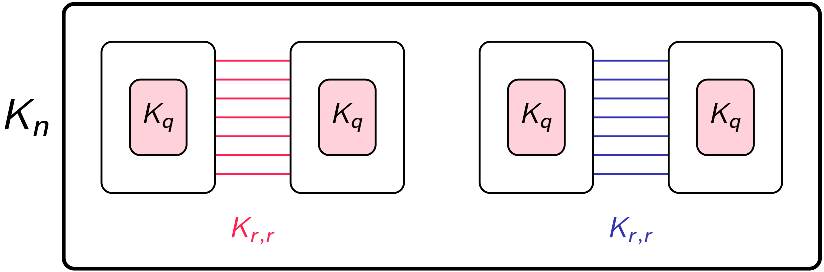

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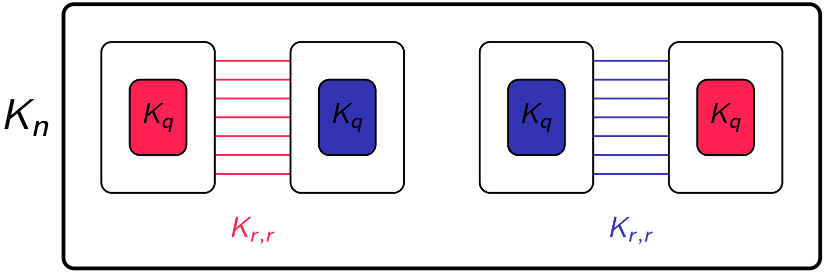

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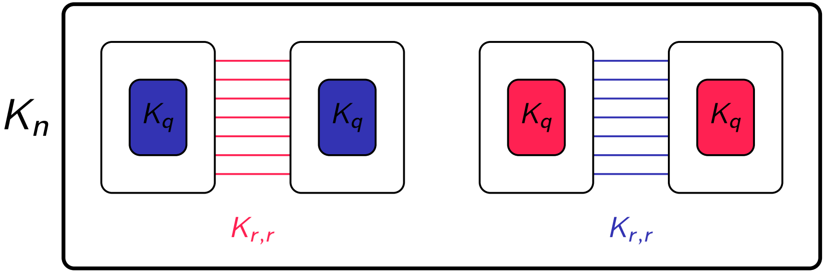

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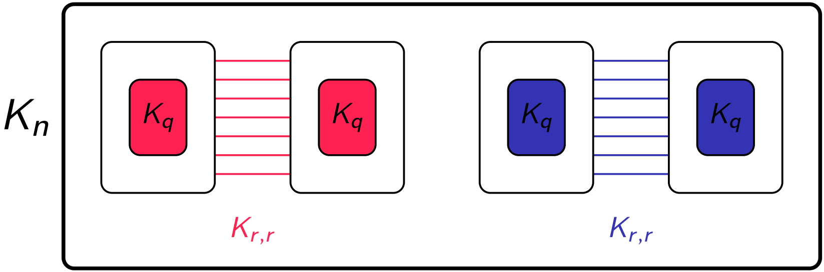

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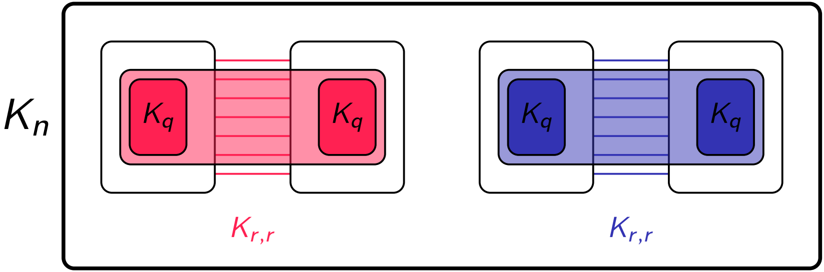

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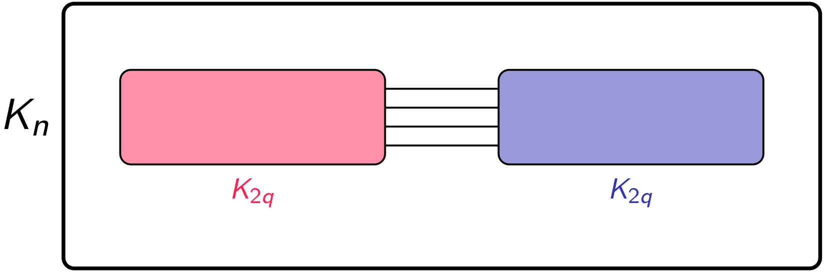

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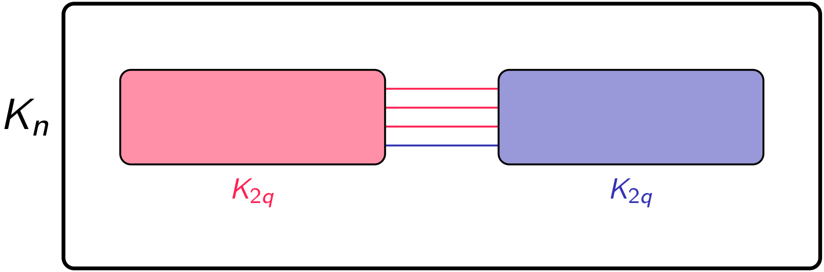

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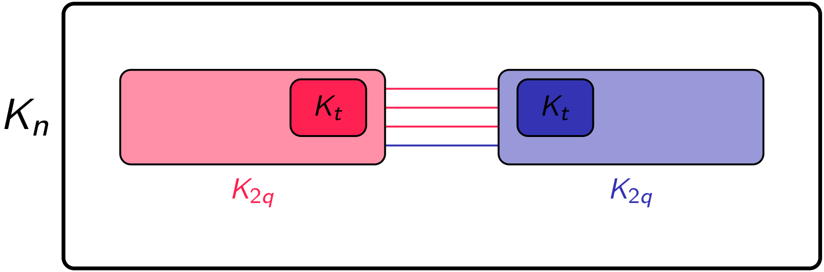

Previous results on balanceability

- Caro, Hansberg, Montejano (2019)
  - $\mathrm{bal}(n, K_4) = n-1$ or n (depending on $n \bmod 4$)
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- Caro, Hansberg, Montejano (2020)
  - Trees are balanceable
    For $n \geq 4k$, $\mathrm{bal}(n, T_k) \leq (k-1)n$
  - For k even and $n \geq \max(3, \frac{k^2}{4} + 1)$,
    $\mathrm{bal}(n, K_{1,k}) = \mathrm{bal}(n, K_{1,k+1}) = \left(\frac{k-2}{2}\right)n - \frac{k^2}{8} + \frac{k}{4}$
  - For $n \geq \frac{9}{32}k^2 + \frac{1}{4}k + 1$,
    $\mathrm{bal}(n, P_{4k}) = \mathrm{bal}(n, P_{4k+1}) = (k-1)n - \frac{1}{2}(k^2 - k - \frac{1}{2})$
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- Caro, Lauri, Zarb (2020)
  - Balancing numbers of the graphs with at most 4 edges

Cycles

Theorem (D., Eslava, Hansberg, Ventura, 2020+)

Let k be a positive integer, and n be an integer such that $n \geq \frac{9}{2}k^2 + \frac{13}{4}k + \frac{49}{32}$, et $\epsilon \in \{-1, 1\}$.

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- $C_{4k+\epsilon}$ is balanceable
- C_{4k} is balanceable

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- C_{4k+2} is not balanceable ;
- $C_{4k+\epsilon}$ is balanceable, and $\text{bal}(n, C_{4k+\epsilon}) = (k-1)n - \frac{1}{2}(k^2 - k - 1 - \epsilon)$;
- C_{4k} is balanceable

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- C_{4k+2} is not balanceable ;
- $C_{4k+\epsilon}$ is balanceable, and $\mathrm{bal}(n, C_{4k+\epsilon}) = (k-1)n - \frac{1}{2}(k^2 - k - 1 - \epsilon)$;
- C_{4k} is balanceable, and $(k-1)n - (k-1)^2 \leq \mathrm{bal}(n, C_{4k}) \leq (k-1)n + 12k^2 + 3k$.

Cycles C_{4k+2}

Proposition

The cycle C_{4k+2} is not balanceable.

Proof by contradiction

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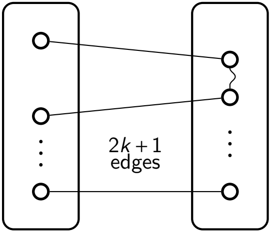

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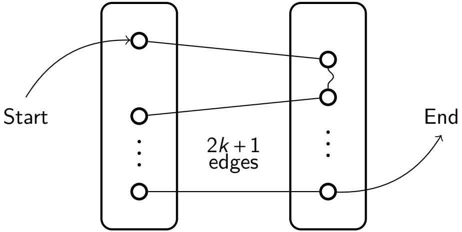

Odd cycles

Proposition

Let k be a positive integer, n be an integer such that $n \geq \frac{9}{2}k^2 + \frac{13}{4}k + \frac{49}{32}$, et $\epsilon \in \{-1, 1\}$.

$$\begin{aligned} \mathrm{bal}(n, C_{4k+\epsilon}) &= \mathrm{bal}(n, P_{4k+\epsilon-1}) \\ &= (k-1)n - \frac{1}{2}(k^2 - k - 1 - \epsilon) \end{aligned}$$

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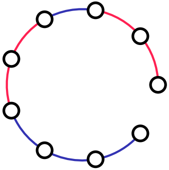

Balanced P_{4k} $\Rightarrow$
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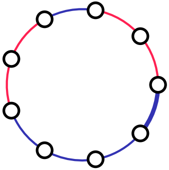

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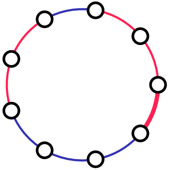

Balanced $P_{4k} \Rightarrow$
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We can close the cycle
which will be balanced

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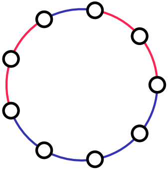
A color with $2k+1$ edges



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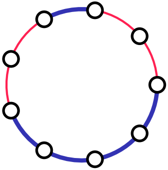

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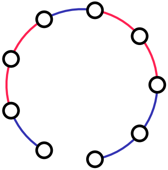
Removing one gives
a balanced P_{4k}



Cycles C_{4k}

The proof for odd cycles does not work:

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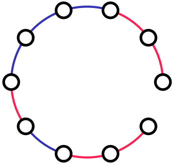

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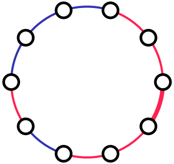

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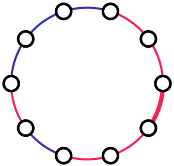

Theorem (D., Eslava, Hansberg, Ventura, 2020+)

For $n \geq \frac{9}{2}k^2 + \frac{13}{4}k + \frac{49}{32}$,

$$(k-1)n - (k-1)^2 \leq \mathrm{bal}(n, C_{4k}) \leq (k-1)n + 12k^2 + 3k$$

Cycles C_{4k}: lower bound

Proposition

For every $n \geq 4k$, $\mathrm{bal}(n, C_{4k}) \geq (k-1)n - (k-1)^2$.

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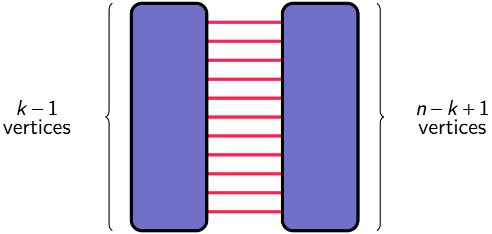

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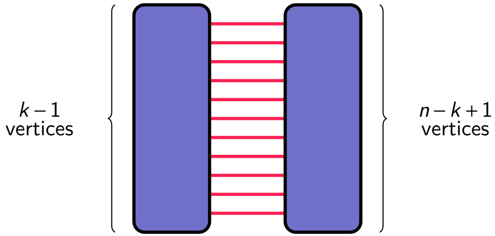

$\Rightarrow$ A cycle can have at most $2k-2$ edges in R.

Cycles C_{4k}: upper bound (1)

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Let $k > 0$ and $n \geq \frac{9}{2}k^2 + \frac{13}{4}k + \frac{49}{32}$:

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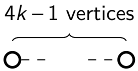

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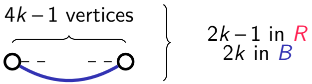
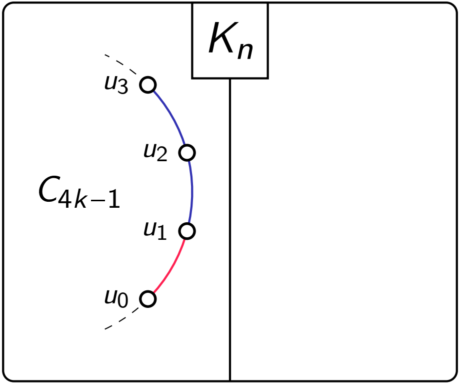
$\Rightarrow \quad$ We close it with (wlog) a B edge



Cycles C_{4k}: upper bound (2)

Proof by contradiction (sequel)



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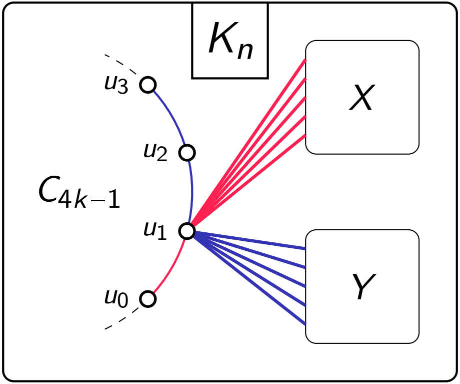

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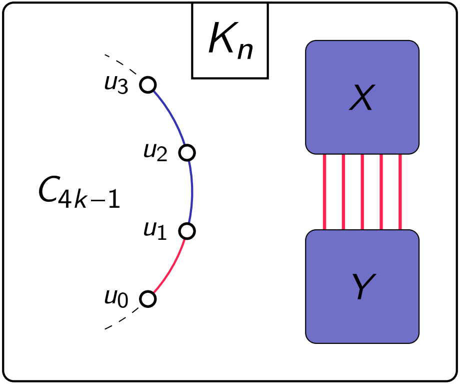

Lemmas enforce the colors of $E(X)$, $E(Y)$ and $E(X, Y)$.

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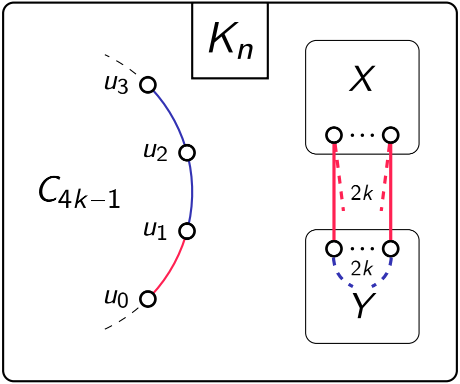

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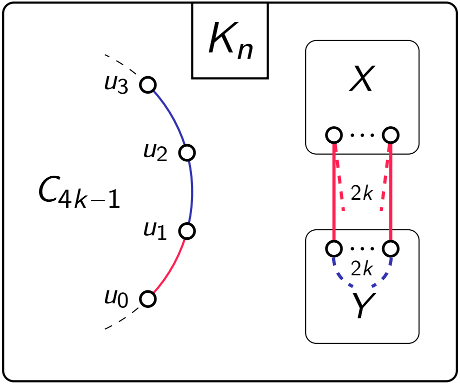

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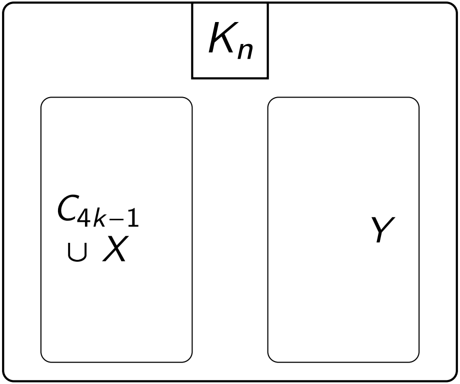

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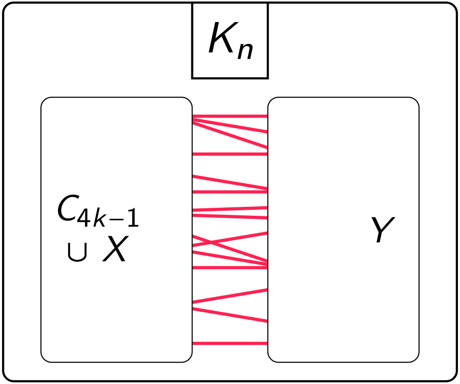

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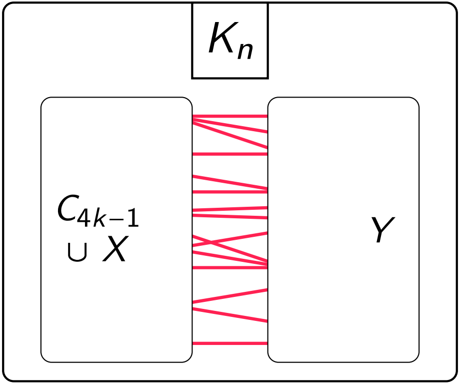

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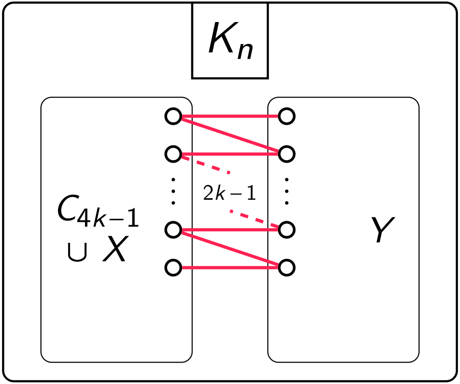

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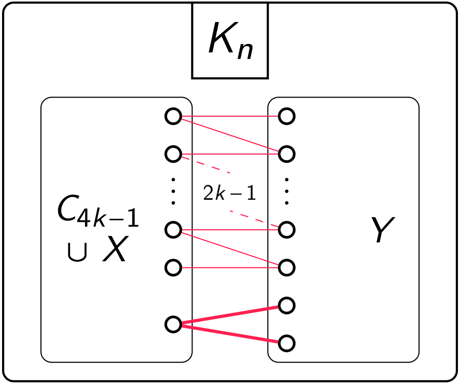

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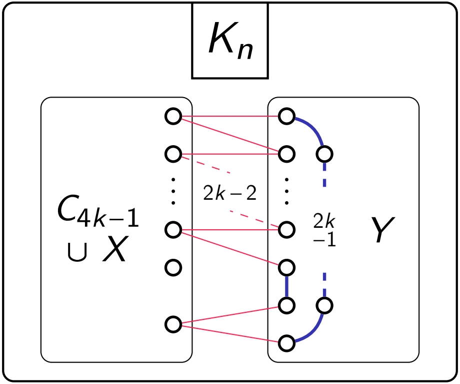

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We complete with edges in Y, which will be in B, and we get a balanced C_{4k}.

$\Rightarrow$ Contradiction

Circulant graphs $C_{k,\ell}$

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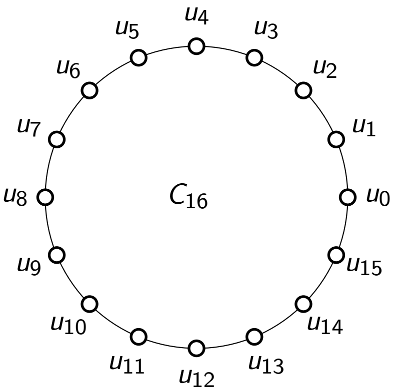
Definition

$C_{k,\ell}$ is a cycle C_k



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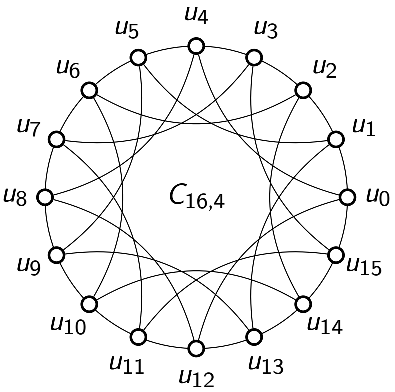

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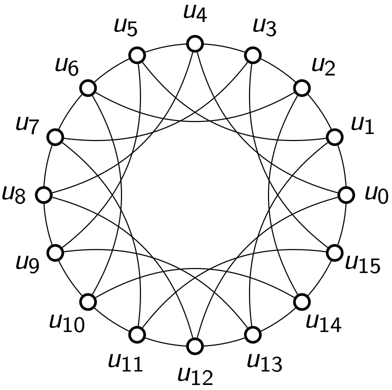

Contains antiprisms and Möbius ladders.

Circulant graphs $C_{k,\ell}$

Theorem (D., Hansberg, Ventura, 2021)

Let $k > 3$ and $\ell \in \{2, \ldots, k-2\}$. The graph $C_{k,\ell}$ is balanceable if and only if k is even and $(k, \ell) \neq (6, 2)$.

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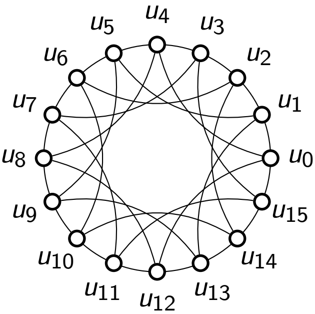

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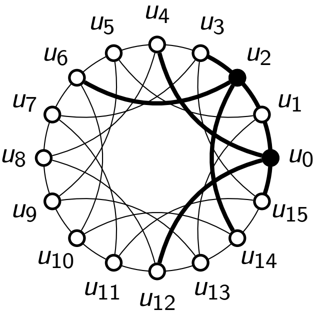

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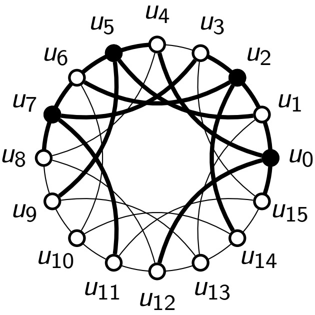

Work in progress

- K_n with $\frac{n(n-1)}{2}$ odd
  Common integer solutions of $k(n-k) = \frac{1}{2}\binom{n}{2} \pm \frac{1}{2}$ and $\binom{\ell}{2} = \frac{1}{2}\binom{n}{2} \pm \frac{1}{2}$

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  1. $n \in \{2, 3, 7, 11, 14, 38, 62, 79, 359, 43.262\} \Rightarrow$ Balanceable
  2. Other values of $n \leq 10^{765.500} \Rightarrow$ Non-balanceable

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$\Rightarrow$ Allows us to break graph operators (disjoint union, joint...)!

Summary

Balanceable

Non-balanceable

exact value of bal(n, G)

non-trivial bounds for bal(n, G)

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Balanceable

Non-balanceable

exact value of $\mathrm{bal}(n, G)$

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Circulant graphs

K_n

- $n \in \{2, 3, 4\}$
- $n \in \{7, 11, 14, 38, 62, 79, 359, 43.262\}$
- $n \geq 5$ and $\frac{n(n-1)}{2}$ even, $n \leq 10^{765.500}$

[CHM, 2020], [CLZ, 2020]
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Cycles

- Odd
- C_{4k}
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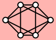
- $C_{6,2}$ 
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Summary

Balanceable | Non-balanceable

exact value of $\mathrm{bal}(n, G)$ | non-trivial bounds for $\mathrm{bal}(n, G)$

Circulant graphs

K_n

- $n \in \{2, 3, 4\}$
- $n \in \{7, 11, 14, 38, 62, 79, 359, 43.262\}$
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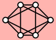
- $C_{6,2}$ 
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Grids

- $G_{2k,2\ell}$
- $G_{2k+1,2\ell+1}$
- T_{8k}, T_{8k+1}
- T_{8k+4}, T_{8k+5}

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Non-balanceable graphs

Theorem (Caro, Hansberg, Montejano, 2020)

A graph is balanceable if and only if it has both:

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Generalized balancing number

Idea

From a 2-edge-coloring to a 2-edge-covering:

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Definition (D., Eslava, Hansberg, Ventura, 2020+)

Let $\text{bal}^*(n,G)$ be the smallest integer such that every 2-edge-covering $R \cup B$ of K_n verifying $|R|, |B| > \text{bal}^*(n,G)$ contains a balanced copy of G.
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$\Rightarrow$ Every graph has a generalized balancing number!

First results

Proposition

If $\mathrm{bal}(n, G)$ exists, then $\mathrm{bal}^*(n, G) = \mathrm{bal}(n, G)$.
Otherwise, $\frac{1}{2}\binom{n}{2} \le \mathrm{bal}^*(n, G) < \binom{n}{2}$.

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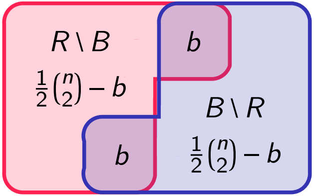

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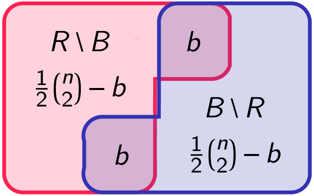

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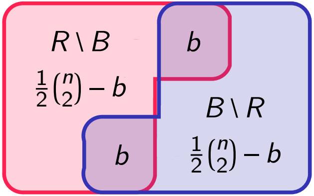

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Proposition

If k bicolored edges guarantee a balanced copy of G, then $\mathrm{bal}^*(n,G) \le \frac{1}{2}\binom{n}{2} + \left\lceil \frac{k}{2} \right\rceil - 1$.

A general upper bound

- $\mathscr{H}(G) = \left\{ H \leq G \mid e(H) = \left\lfloor \frac{e(G)}{2} \right\rfloor,\ H \text{ with no isolated vertex} \right\}$

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For every $G(V,E)$ and $n \geq |V|$, we have

$$\text{bal*}(n,G) \leq \frac{1}{2}\binom{n}{2} + \left\lceil \frac{\text{ex}(n,\mathscr{H}(G))}{2} \right\rceil.$$

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Proof

If there are at least $\text{ex}(n,\mathcal{H}(G)) + 1$ bicolored edges, we can select them, complete the copy of G, and assign the colors of the bicolored edges to balance the copy.

Applying the upper bound

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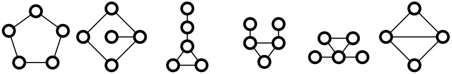
- $\mathscr{H}(K_5) =$ 

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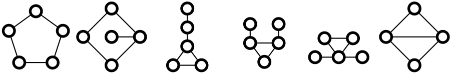
- $\mathscr{H}(K_5) =$ 

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For every integer $n \geq 5$, $\mathrm{ex}(n, \mathscr{H}(K_5)) = \mathrm{ex}(n, \{C_3, C_4, C_5\})$.

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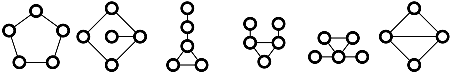
- $\mathscr{H}(K_5) =$ 

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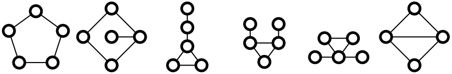
- $\mathscr{H}(K_5) =$ 

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$\rightarrow$ Quality of this bound?

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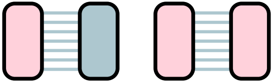
type A  type B

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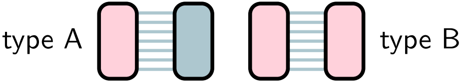

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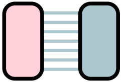
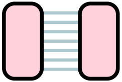
type A   type B

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4. If type B copy: wherever the bicolored edge is, we can find a balanced copy of C_{4k+2}.

Lower bound for K_5

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Proposition

Let $c = \left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{\frac{5}{2}}$. We have $\text{bal}^*(n, K_5) \geq \frac{1}{2}\binom{n}{2} + (1-\epsilon)cn^{\frac{3}{2}}$.

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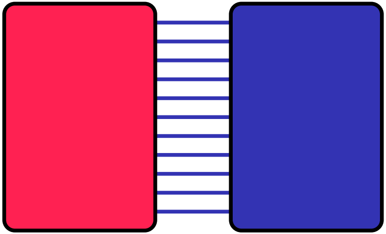

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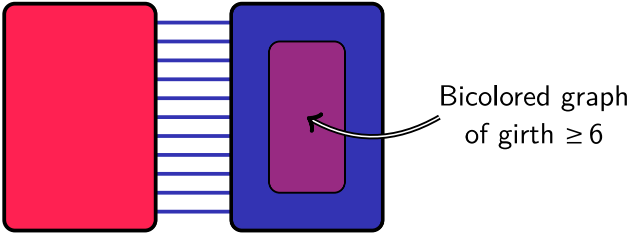

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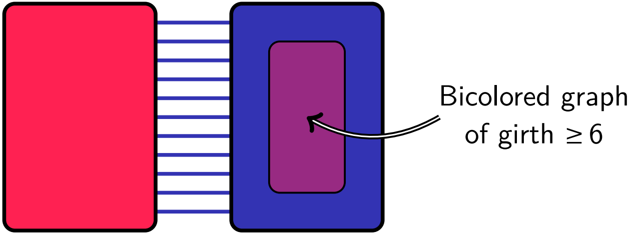

Then, we prove $|R|, |B| > \frac{1}{2}\binom{n}{2} + (1-\epsilon)cn^{\frac{3}{2}}$.

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$\rightarrow$ There are differences among non-balanceable graphs.

Final words

Conclusion

- Balanceability results, study of $\mathrm{bal}(n, G)$
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Open questions

- Complexity
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