# Balancing graphs using bicolored edges

Antoine Dailly

Joint work with Adriana Hansberg and Denae Ventura.

# Recall the context

**2-coloring**

A 2-coloring of the edges of $K_n$ is a partition $E(K_n) = R \cup B$.

**Balanced copy**

Within a 2-coloring of the edges of $K_n$, a balanced copy of $G$ is a copy of $G$ with half its edges in $R$ and the other half in $B$.

# Recall the context

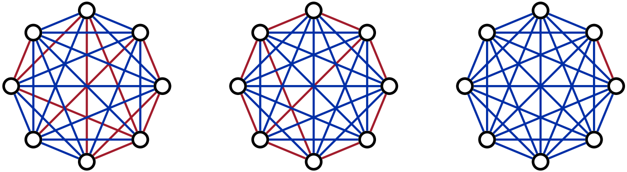
**2-coloring**

A 2-coloring of the edges of $K_n$ is a partition $E(K_n) = R \cup B$.

**Balanced copy**

Within a 2-coloring of the edges of $K_n$, a balanced copy of $G$ is a copy of $G$ with half its edges in $R$ and the other half in $B$.

Example: finding a balanced copy of $C_4$



# Recall the context

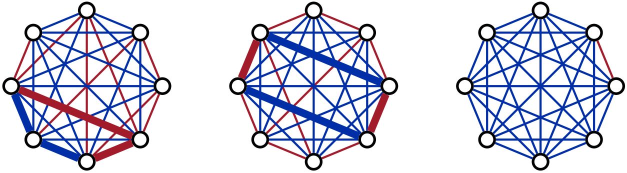
**2-coloring**

A 2-coloring of the edges of $K_n$ is a partition $E(K_n) = R \cup B$.

**Balanced copy**

Within a 2-coloring of the edges of $K_n$, a balanced copy of $G$ is a copy of $G$ with half its edges in $R$ and the other half in $B$.

Example: finding a balanced copy of $C_4$



Impossible: only one edge in $R$.

# Balanceability

**Definition**

A graph $G$ is balanceable if there exists a $k$ such that, for $n$ sufficiently large, every 2-coloring of the edges of $K_n$ with $|R|, |B| > k$ contains a balanced copy of $G$.

# Balanceability

**Definition**

A graph $G$ is balanceable if there exists a $k$ such that, for $n$ sufficiently large, every 2-coloring of the edges of $K_n$ with $|R|, |B| > k$ contains a balanced copy of $G$.
The smallest $k$ such that this holds is the balancing number, denoted by $\text{bal}(n, G)$.

# Balanceability

**Definition**

A graph $G$ is balanceable if there exists a $k$ such that, for $n$ sufficiently large, every 2-coloring of the edges of $K_n$ with $|R|, |B| > k$ contains a balanced copy of $G$.
The smallest $k$ such that this holds is the balancing number, denoted by $\text{bal}(n, G)$.

## Example

$\text{bal}(n, C_4) = 1$

- $\geq 1$ by the previous slide;
- Easy to check that at least 2 edges of each color are enough to find a balanced $C_4$.

# Characterization

**Theorem** (Caro, Hansberg, Montejano, 2019+)

A graph is balanceable if and only if it has both:

- A cut crossed by half its edges;
- An induced subgraph containing half its edges.

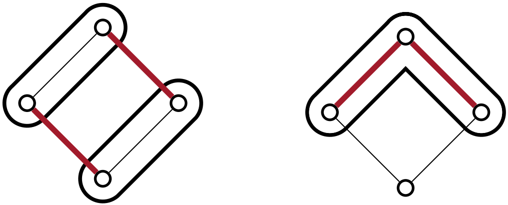
# Characterization

**Theorem** (Caro, Hansberg, Montejano, 2019+)

A graph is balanceable if and only if it has both:

- A cut crossed by half its edges;
- An induced subgraph containing half its edges.

Example: $C_4$ is balanceable



# Sufficient condition for non-balanceability

**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

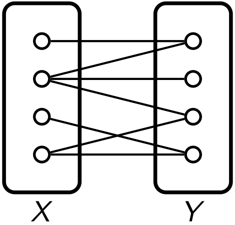
# Sufficient condition for non-balanceability

**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

## Proof by contradiction

Let $(X, Y)$ be a cut crossed by half the edges



# Sufficient condition for non-balanceability

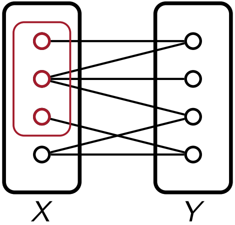
**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

Proof by contradiction

Let $(X, Y)$ be a cut crossed by half the edges

$X_{odd}$: set of vertices of odd cut-degree in $X$



# Sufficient condition for non-balanceability

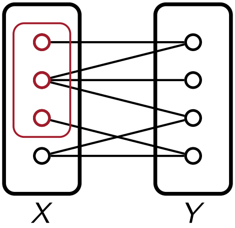
**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

## Proof by contradiction

Let $(X, Y)$ be a cut crossed by half the edges

$X_{odd}$: set of vertices of odd cut-degree in $X$




$|X_{odd}|$ is odd since $\frac{|E|}{2}$ is odd.

# Sufficient condition for non-balanceability

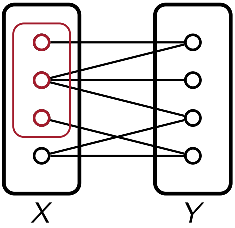
**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

## Proof by contradiction

Let $(X, Y)$ be a cut crossed by half the edges

$X_{odd}$: set of vertices of odd cut-degree in $X$




$|X_{odd}|$ is odd since $\frac{|E|}{2}$ is odd. Also, vertices in $X_{odd}$ have odd degree in $G[X]$.

# Sufficient condition for non-balanceability

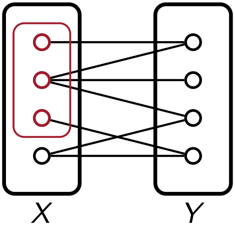
**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

## Proof by contradiction

Let $(X, Y)$ be a cut crossed by half the edges

$X_{odd}$: set of vertices of odd cut-degree in $X$




$|X_{odd}|$ is odd since $\frac{|E|}{2}$ is odd. Also, vertices in $X_{odd}$ have odd degree in $G[X]$. $\Rightarrow$ Sum of degrees in $G[X]$ is odd $\Rightarrow$ Contradiction

# Non-balanceable graphs

**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

Corollary

$K_{8k+5}$ is not balanceable.

# Non-balanceable graphs

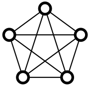
**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

### Corollary

$K_{8k+5}$ is not balanceable.

$\rightarrow$ Question: "how unbalanceable" is $K_5$?



# Non-balanceable graphs

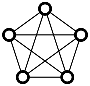
**Proposition**

If $G$ is eulerian and $\frac{|E(G)|}{2}$ is odd, then $G$ is not balanceable.

## Corollary

$K_{8k+5}$ is not balanceable.

$\rightarrow$ Question: "how unbalanceable" is $K_5$?



How hard would it be to guarantee a balanced copy of $K_5$ by relaxing the problem?

# Introducing bicolored edges

**Bicolored edges**

Let $E(K_n) = R \cup B$ be a 2-coloring of the edges of $K_n$. We now allow $R \cap B \neq \emptyset$.

# Introducing bicolored edges

**Bicolored edges**

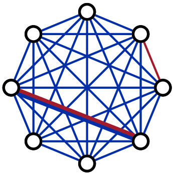
Let $E(K_n) = R \cup B$ be a 2-coloring of the edges of $K_n$. We now allow $R \cap B \neq \emptyset$.
An edge in $R \cap B$ is called bicolored, and we can set it of either color as needed.

# Introducing bicolored edges

**Bicolored edges**

Let $E(K_n) = R \cup B$ be a 2-coloring of the edges of $K_n$. We now allow $R \cap B \neq \emptyset$.
An edge in $R \cap B$ is called bicolored, and we can set it of either color as needed.

Example with $C_4$

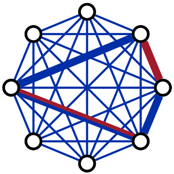


# Introducing bicolored edges

**Bicolored edges**

Let $E(K_n) = R \cup B$ be a 2-coloring of the edges of $K_n$. We now allow $R \cap B \neq \emptyset$.
An edge in $R \cap B$ is called bicolored, and we can set it of either color as needed.

## Example with $C_4$



# Redefining the balancing number

**Balancing number: extension**

For a graph $G$, we call $\mathrm{bal}(n, G)$ the smallest $k$ such that every 2-coloring $R \cup B$ of the edges of $K_n$ allowing bicolored edges with $|R|, |B| > k$ contains a balanced copy of $G$.

Now, even non-balanceable graphs have a balancing number!

# Redefining the balancing number

**Balancing number: extension**

For a graph $G$, we call $\mathrm{bal}(n, G)$ the smallest $k$ such that every 2-coloring $R \cup B$ of the edges of $K_n$ allowing bicolored edges with $|R|, |B| > k$ contains a balanced copy of $G$.

Now, even non-balanceable graphs have a balancing number!

## First observations

- If $G$ is balanceable, then $\mathrm{bal}(n, G)$ does not change.

# Redefining the balancing number

**Balancing number: extension**

For a graph $G$, we call $\mathrm{bal}(n, G)$ the smallest $k$ such that every 2-coloring $R \cup B$ of the edges of $K_n$ allowing bicolored edges with $|R|, |B| > k$ contains a balanced copy of $G$.

Now, even non-balanceable graphs have a balancing number!

## First observations

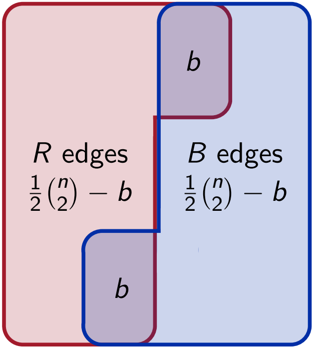
- If $G$ is balanceable, then $\mathrm{bal}(n, G)$ does not change.
- If $G$ is not balanceable, then $\mathrm{bal}(n, G) \geq \frac{1}{2}\binom{n}{2}$ since we need bicolored edges.
- $\mathrm{bal}(n, G) < \binom{n}{2}$: if $R = B = E(K_n)$ then we will find a balanced copy of $G$.

# Counting the bicolored edges

Assume that $|R|, |B| = \frac{1}{2}\binom{n}{2} + b$.

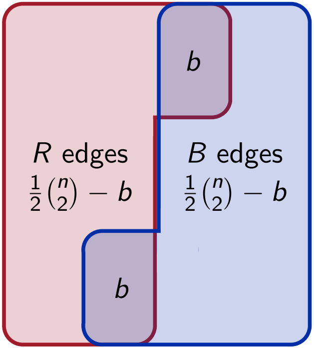
# Counting the bicolored edges

Assume that $|R|, |B| = \frac{1}{2}\binom{n}{2} + b$.



# Counting the bicolored edges

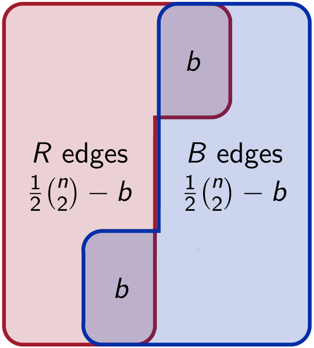
Assume that $|R|, |B| = \frac{1}{2}\binom{n}{2} + b$.




$\Rightarrow 2b$ bicolored edges

# Counting the bicolored edges

Assume that $|R|, |B| = \frac{1}{2}\binom{n}{2} + b$.




$\Rightarrow 2b$ bicolored edges

**Proposition**

If having $k$ bicolored edges guarantees a balanced copy of $G$, then $\mathrm{bal}(n, G) \leq \frac{1}{2}\binom{n}{2} + \lceil \frac{k}{2} \rceil - 1$.

# An upper bound

**Turán number of a family $\mathcal{H}$**

ex$(n, \mathcal{H})$ is the maximum number of edges in a graph of order $n$ and containing no (not necessarily induced) $H$ for $H \in \mathcal{H}$.

# An upper bound

**Turán number of a family $\mathcal{H}$**

$\text{ex}(n, \mathcal{H})$ is the maximum number of edges in a graph of order $n$ and containing no (not necessarily induced) $H$ for $H \in \mathcal{H}$.

**Theorem: general upper bound**

Let $G$ be a graph, and $\mathcal{H}$ be the family of subgraphs of $G$ with $\geq \frac{|E(G)|}{2}$ edges. We have $\text{bal}(n, G) \leq \frac{1}{2}\binom{n}{2} + \lceil \frac{\text{ex}(n,\mathcal{H})}{2} \rceil$.

# An upper bound

**Turán number of a family $\mathcal{H}$**

$\mathrm{ex}(n, \mathcal{H})$ is the maximum number of edges in a graph of order $n$ and containing no (not necessarily induced) $H$ for $H \in \mathcal{H}$.

**Theorem: general upper bound**

Let $G$ be a graph, and $\mathcal{H}$ be the family of subgraphs of $G$ with $\geq \frac{|E(G)|}{2}$ edges. We have $\mathrm{bal}(n, G) \leq \frac{1}{2}\binom{n}{2} + \lceil \frac{\mathrm{ex}(n,\mathcal{H})}{2} \rceil$.

## Proof idea

If $|R|, |B| > \frac{1}{2}\binom{n}{2} + \lceil \frac{\mathrm{ex}(n,\mathcal{H})}{2} \rceil$, then there are at least $\mathrm{ex}(n, \mathcal{H}) + 1$ bicolored edges.

# An upper bound

**Turán number of a family $\mathcal{H}$**

$\text{ex}(n, \mathcal{H})$ is the maximum number of edges in a graph of order $n$ and containing no (not necessarily induced) $H$ for $H \in \mathcal{H}$.

**Theorem: general upper bound**

Let $G$ be a graph, and $\mathcal{H}$ be the family of subgraphs of $G$ with $\geq \frac{|E(G)|}{2}$ edges. We have $\text{bal}(n, G) \leq \frac{1}{2}\binom{n}{2} + \lceil \frac{\text{ex}(n,\mathcal{H})}{2} \rceil$.

## Proof idea

If $|R|, |B| > \frac{1}{2}\binom{n}{2} + \lceil \frac{\text{ex}(n,\mathcal{H})}{2} \rceil$, then there are at least $\text{ex}(n, \mathcal{H}) + 1$ bicolored edges. Select bicolored edges forming a graph in $\mathcal{H}$, add edges to get $G$, and set the bicolored edges as needed to have a balanced copy.

# Applying the theorem

**Upper bound for** $\text{bal}(n, K_5)$

$$\text{bal}(n, K_5) \leq \frac{1}{2}\binom{n}{2} + \left\lceil \frac{\text{ex}(n,\{C_3,C_4,C_5\})}{2} \right\rceil$$

# Applying the theorem

**Upper bound for** $\mathrm{bal}(n, K_5)$

$$\mathrm{bal}(n, K_5) \leq \frac{1}{2}\binom{n}{2} + \left\lceil \frac{\mathrm{ex}(n, \{C_3, C_4, C_5\})}{2} \right\rceil$$

## Proof idea

A graph with more than $\mathrm{ex}(n, \{C_3, C_4, C_5\})$ edges has girth $\leq 5$ and a certain density.

$\Rightarrow$ It has at least 5 edges among 5 vertices.

$\Rightarrow$ We can apply the theorem.

# Applying the theorem

**Upper bound for $\mathrm{bal}(n, K_5)$**

$$\mathrm{bal}(n, K_5) \leq \frac{1}{2}\binom{n}{2} + \left\lceil \frac{\mathrm{ex}(n,\{C_3,C_4,C_5\})}{2} \right\rceil$$

## Proof idea

A graph with more than $\mathrm{ex}(n, \{C_3, C_4, C_5\})$ edges has girth $\leq 5$ and a certain density.
$\Rightarrow$ It has at least 5 edges among 5 vertices.
$\Rightarrow$ We can apply the theorem.

## Question

How good is this upper bound?

# A lower bound for $\mathrm{bal}(n, K_5)$

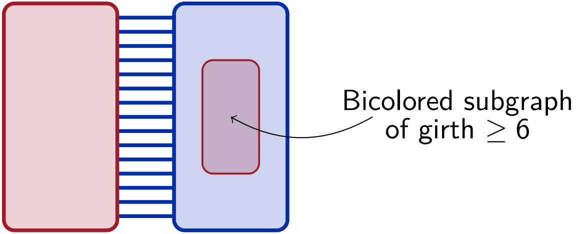
**Lower bound**

There is a 2-coloring $R \cup B$ of the edges of $K_n$ with $|R|, |B| \geq \frac{1}{2}\binom{n}{2} + \theta(\mathrm{ex}(n, \{C_3, C_4, C_5\}))$ without a balanced $K_5$.

# A lower bound for $\mathrm{bal}(n, K_5)$

**Lower bound**

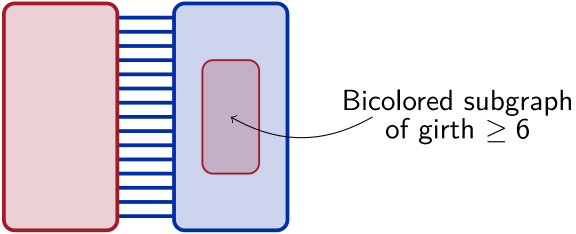
There is a 2-coloring $R \cup B$ of the edges of $K_n$ with $|R|, |B| \geq \frac{1}{2}\binom{n}{2} + \theta(\mathrm{ex}(n, \{C_3, C_4, C_5\}))$ without a balanced $K_5$.



# A lower bound for $\mathrm{bal}(n, K_5)$

**Lower bound**

There is a 2-coloring $R \cup B$ of the edges of $K_n$ with $|R|, |B| \geq \frac{1}{2}\binom{n}{2} + \theta(\mathrm{ex}(n, \{C_3, C_4, C_5\}))$ without a balanced $K_5$.




We can have $|R|, |B| = \frac{1}{2}\binom{n}{2} + \theta(\mathrm{ex}(n, \{C_3, C_4, C_5\}))$.

# Conclusion

**Theorem**

$\mathrm{bal}(n, K_5) = \frac{1}{2}\binom{n}{2} + \theta(\mathrm{ex}(n, \{C_3, C_4, C_5\}))$.

# Conclusion

**Theorem**

$\mathrm{bal}(n, K_5) = \frac{1}{2}\binom{n}{2} + \theta(\mathrm{ex}(n, \{C_3, C_4, C_5\}))$.

So, what does it mean?

$\theta(\mathrm{ex}(n, \{C_3, C_4, C_5\})) = \theta(n^{\frac{3}{2}})$ bicolored edges are both *necessary* and *sufficient* to balance $K_5$.

# Conclusion

**Theorem**

$\mathrm{bal}(n, K_5) = \frac{1}{2}\binom{n}{2} + \theta(\mathrm{ex}(n, \{C_3, C_4, C_5\}))$.

So, what does it mean?

$\theta(\mathrm{ex}(n, \{C_3, C_4, C_5\})) = \theta(n^{\frac{3}{2}})$ bicolored edges are both *necessary* and *sufficient* to balance $K_5$.

However, in the case of $C_{4k+2}$, we need 1 bicolored edge while the theorem for the general upper bound gives an upper bound of $\frac{1}{2}\binom{n}{2} + \theta(nk - k^2)$.

# Conclusion

**Theorem**

$\text{bal}(n, K_5) = \frac{1}{2}\binom{n}{2} + \theta(\text{ex}(n, \{C_3, C_4, C_5\}))$.

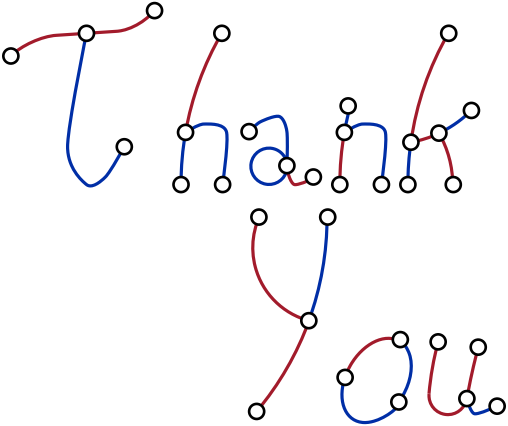
## So, what does it mean?

$\theta(\text{ex}(n, \{C_3, C_4, C_5\})) = \theta(n^{\frac{3}{2}})$ bicolored edges are both *necessary* and *sufficient* to balance $K_5$.

However, in the case of $C_{4k+2}$, we need 1 bicolored edge while the theorem for the general upper bound gives an upper bound of $\frac{1}{2}\binom{n}{2} + \theta(nk - k^2)$.

## Questions

Why this difference? What makes a graph "difficult" to balance, even with bicolored edges?


Thank
You