Octal Games on Graphs : 0.03 and 0.33

Antoine Dailly

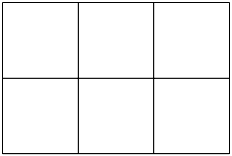
With the ANR GAG (Graphs and Games) Project
(Aline Parreau, Pierre Coupechoux, Éric Sopena, Laurent Beaudou, Sylvain Gravier)

Section 1

Introduction

Let's play !

Game of CRAM: Two players successively place dominos on the board, the last player who is able to place a domino wins the game.



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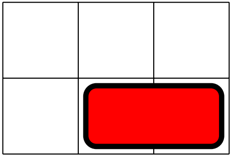

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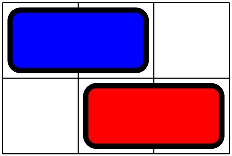

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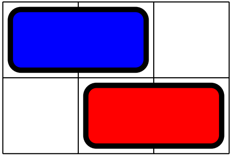

⇒ Second player wins.

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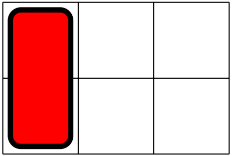

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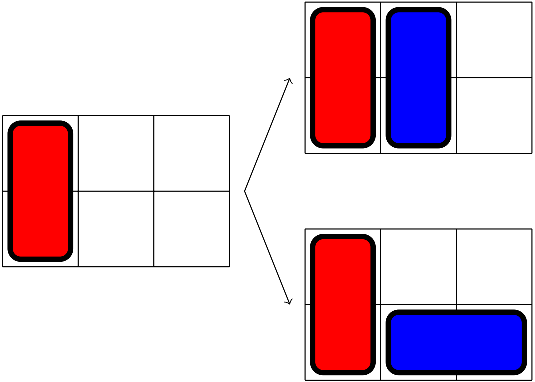

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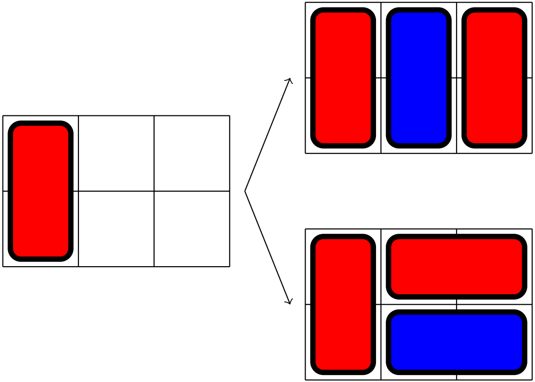

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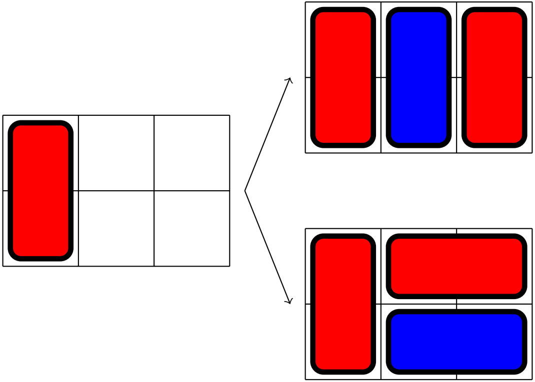
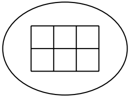


$\Rightarrow$ First player wins.

The game graph



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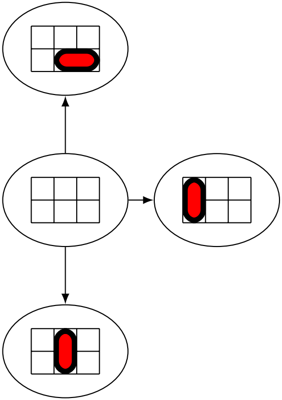

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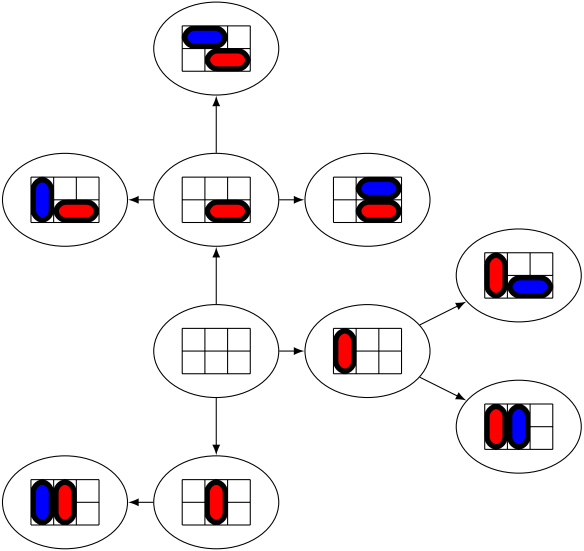

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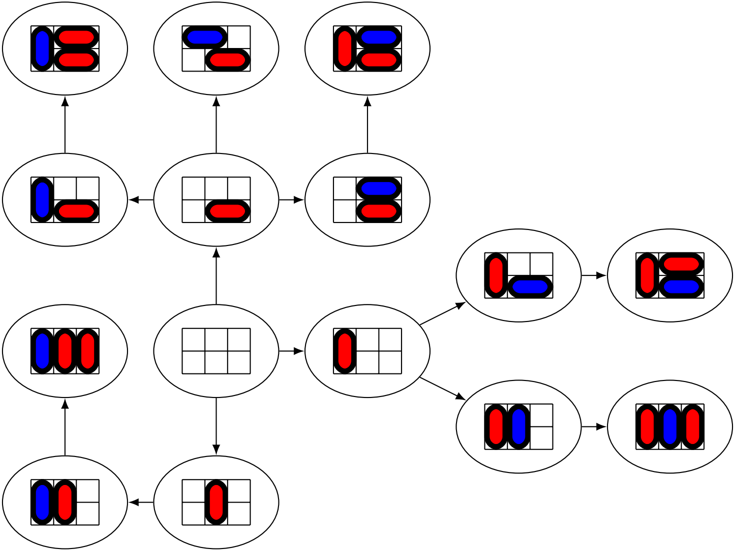

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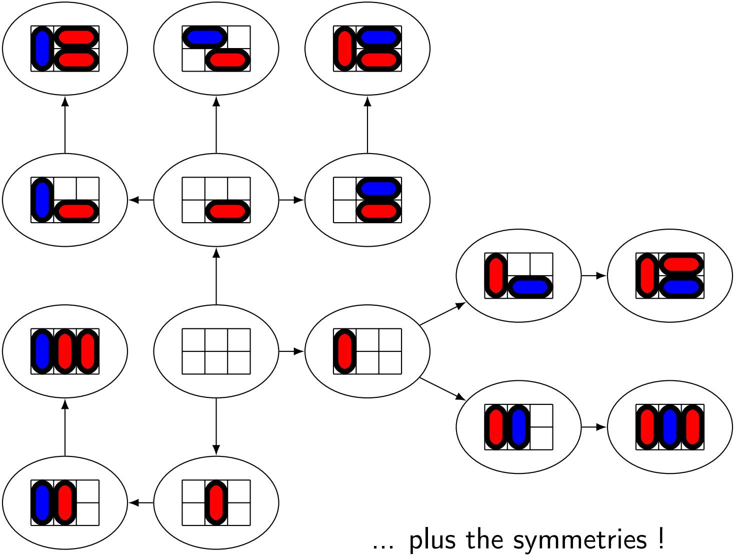

... plus the symmetries !

$\mathcal{N}$ and $\mathcal{P}$

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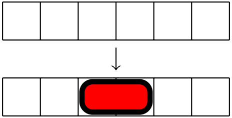

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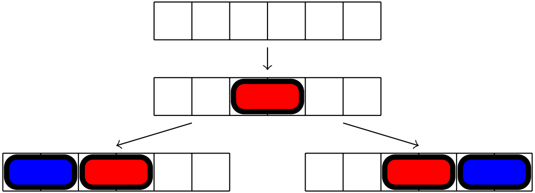

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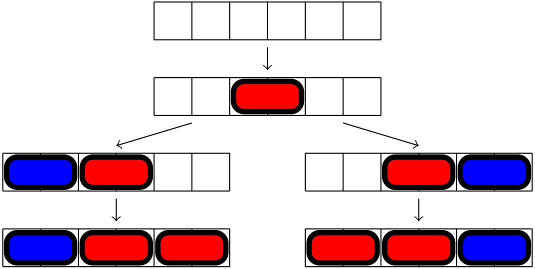

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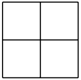

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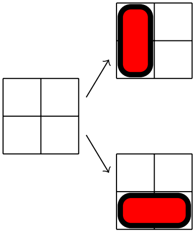

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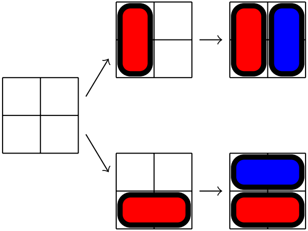

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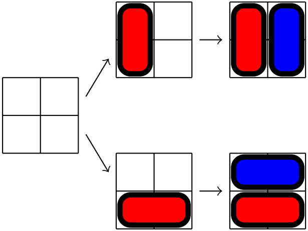
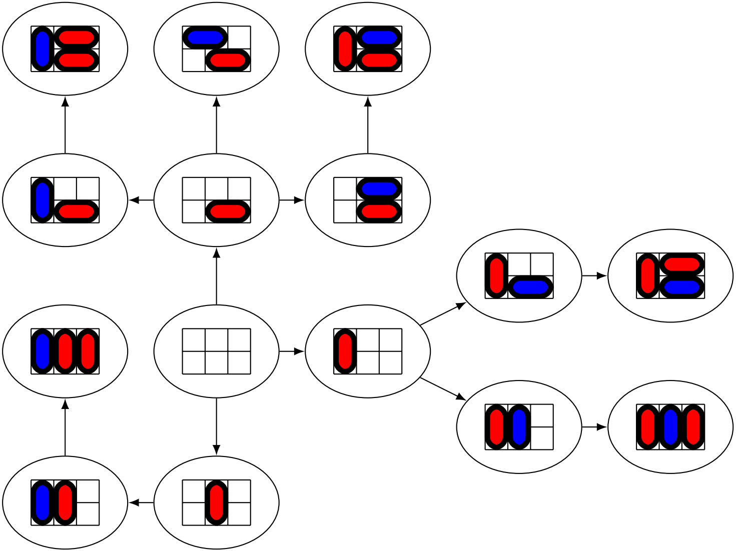


$\Rightarrow$ How to determine if a game is an $\mathcal{N}$ or a $\mathcal{P}$-position ?

Using the game graph



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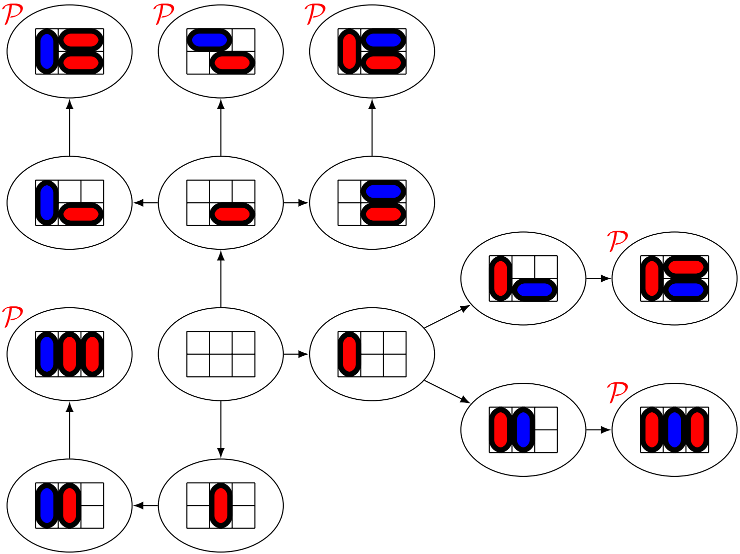
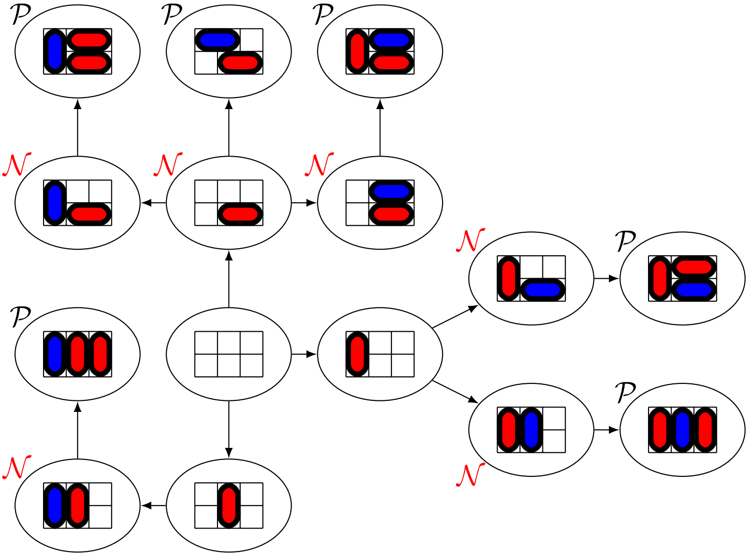


Using the game graph



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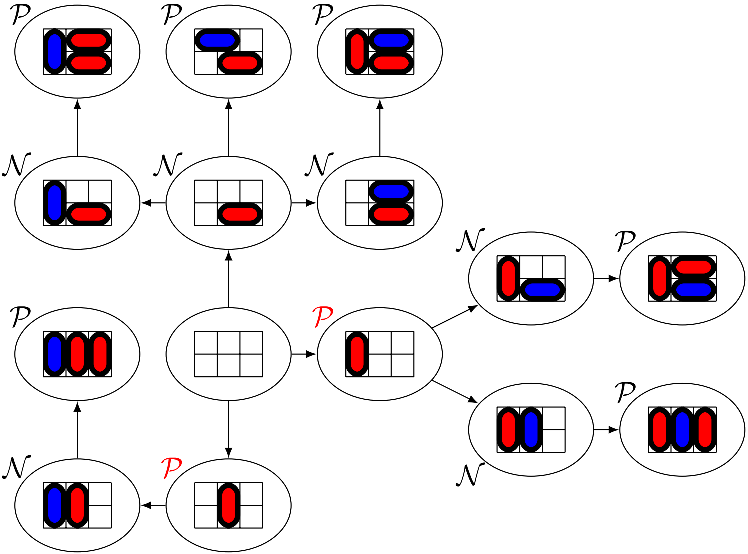

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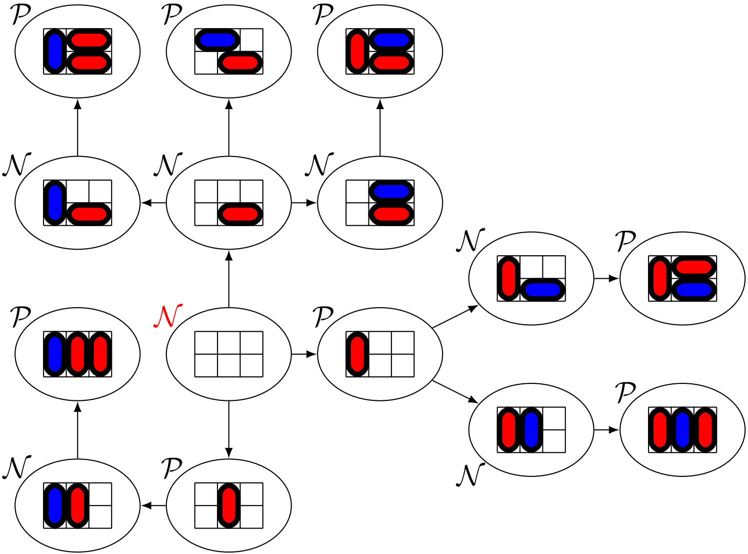
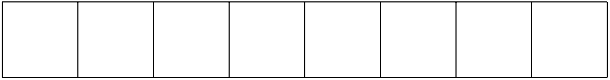


Octal Games

Playing CRAM-like games on a row:



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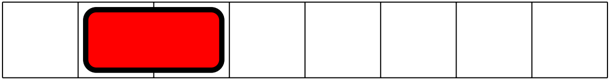

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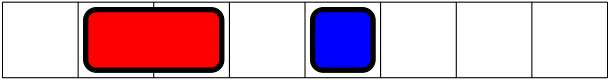

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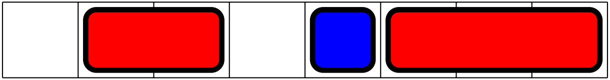

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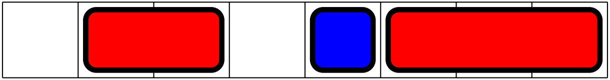

Notation: $0.u_1 u_2 ... u_n ...$

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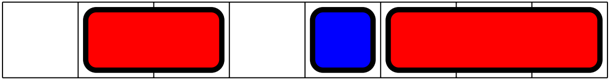

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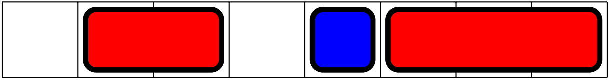

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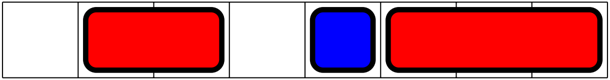

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- CRAM is 0.07

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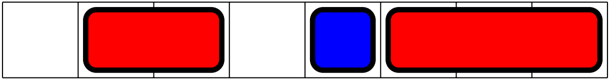

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- CRAM is 0.07
- The game up here could be 0.471

Studying Octal Games

Definition

The *sequence* of an octal game is the string $o_0 o_1 ... o_n ...$ where o_i is the outcome of the game on a line of size i.

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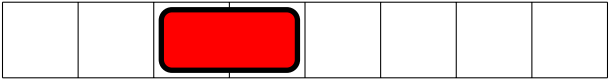
n	0	1	2	3	4	5	6	7	8	9	10
Outcome	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$

Summing Games



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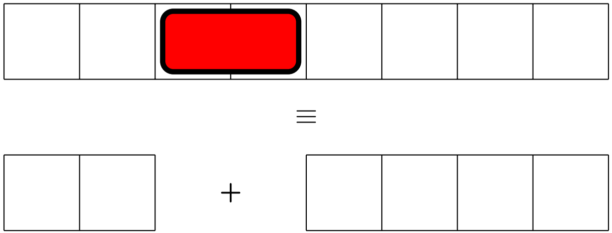

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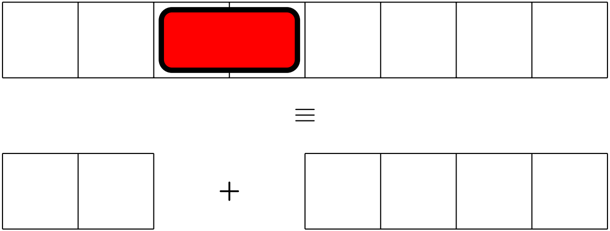

Disjunctive sum

When playing on $G + H$, a player has to play either on G or on H.
The winner is the last one able to play.

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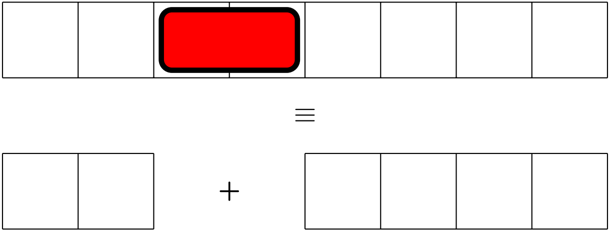

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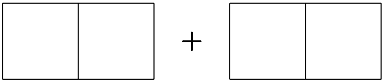
Proposition

If G is a $\mathcal{P}$-position, then, for every game H, $G + H$ has the same outcome than H.

Summing $\mathcal{N}$-positions



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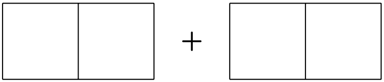

$\Rightarrow \mathcal{P}$

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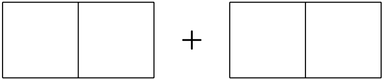

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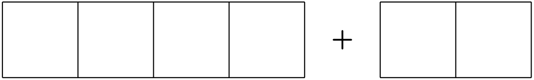

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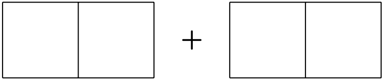

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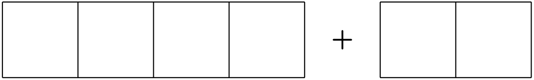

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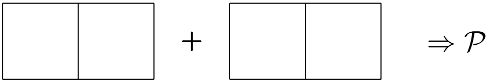

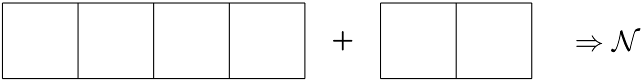

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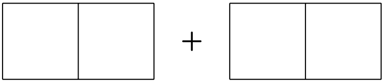

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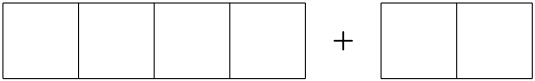

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- Equivalence classes for games, mapped to the positive integers (for $\mathcal{N}$-positions) and 0 (for $\mathcal{P}$-positions)

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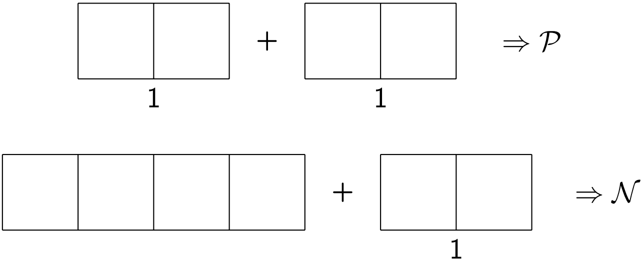

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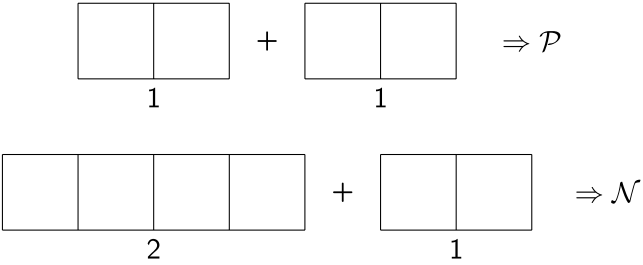

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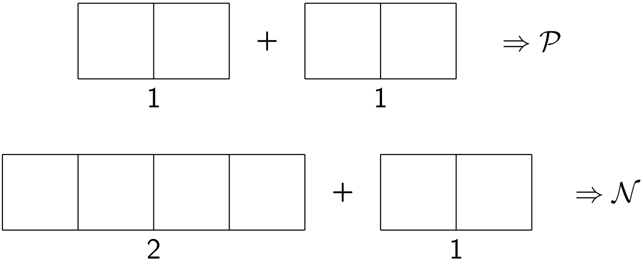

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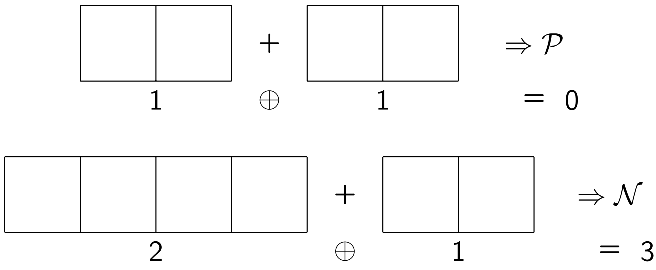

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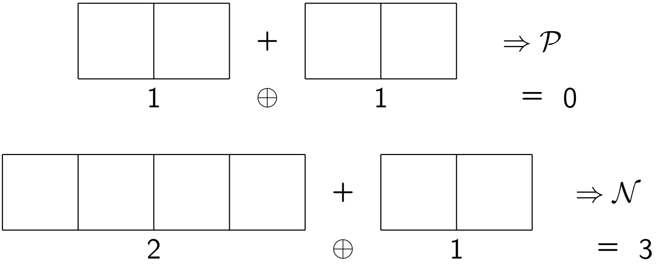

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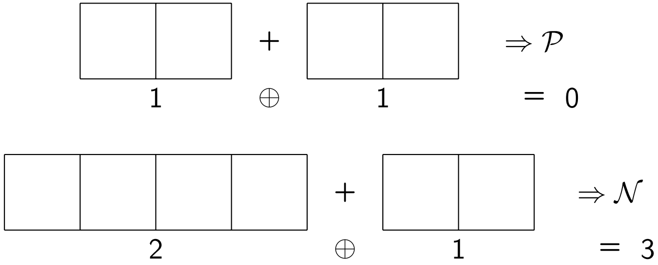
- *Nim-sequence* of an octal game: the sequence of its values.

Summing $\mathcal{N}$-positions



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- Direct computation of the sum is possible: XOR operator
- *Nim-sequence* of an octal game: the sequence of its values.

n	0	1	2	3	4	5	6	7	8	9	10
Value	0	0	1	1	2	0	3	1	1	0	3

Octal Games on Graphs

A chain is similar to a row...

Octal Games on Graphs

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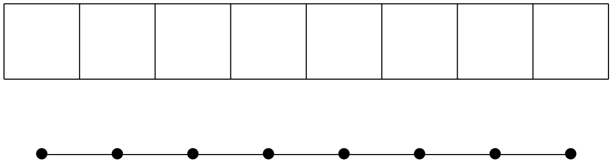

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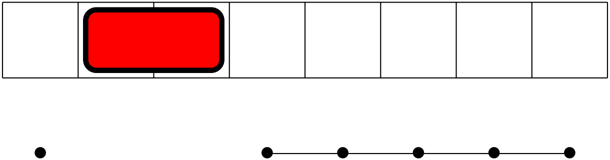

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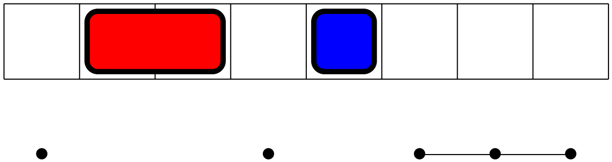

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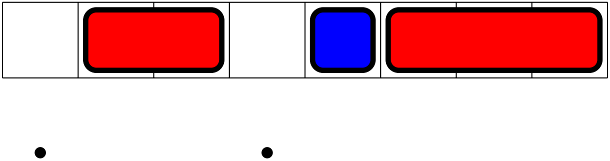

Octal Games on Graphs

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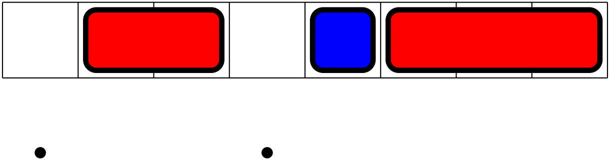

Notation: $0.u_1u_2...u_n...$

- i vertices can be removed from the graph iff $u_i \neq 0$

Octal Games on Graphs

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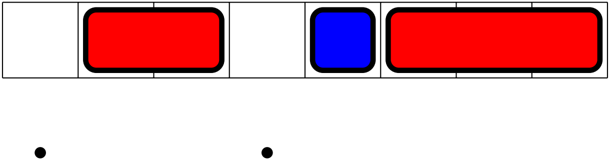

Notation: $0.u_1 u_2 ... u_n ...$

- i vertices can be removed from the graph iff $u_i \neq 0$
- i vertices can be removed in a certain way depending on the value of u_i (for $1 \leq u_i \leq 7$)

Octal Games on Graphs

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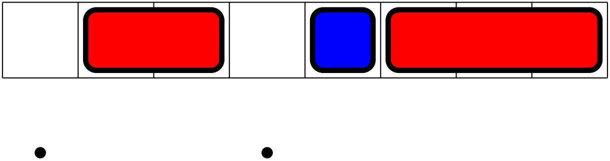

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- The removed vertices must induce a connected subgraph

Octal Games on Graphs

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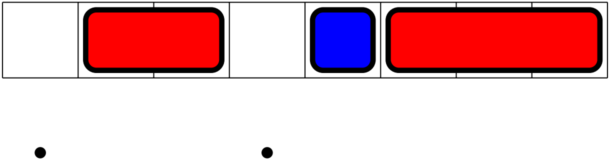

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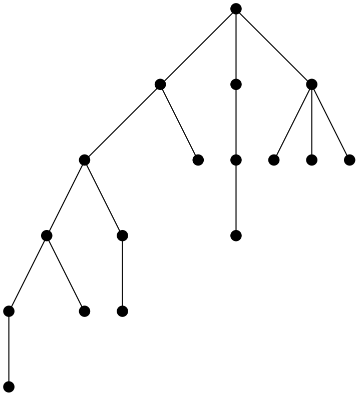
Few results, mostly on 0.07, which is called ARC-KAYLES

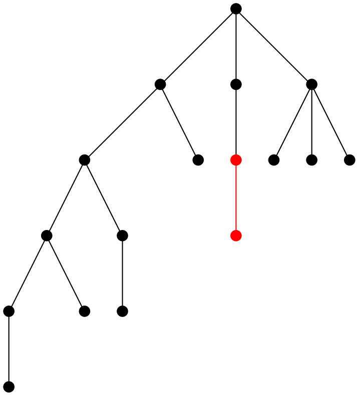
Section 2

0.03 on graphs

0.03 on trees



0.03 on trees



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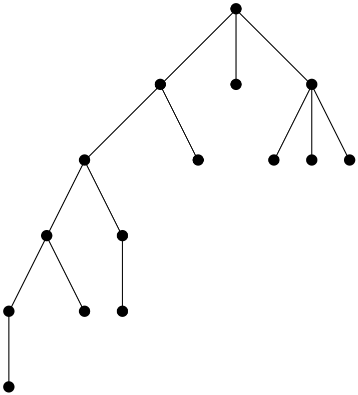

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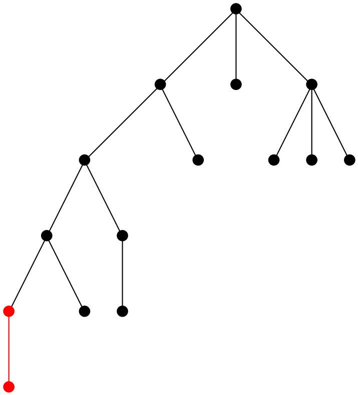

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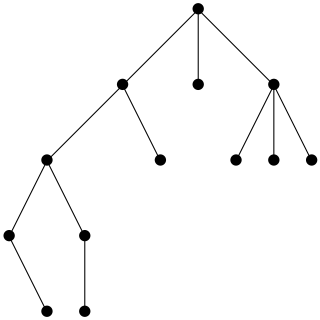

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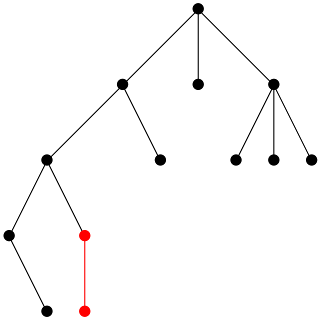

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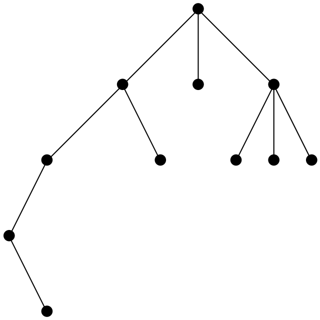

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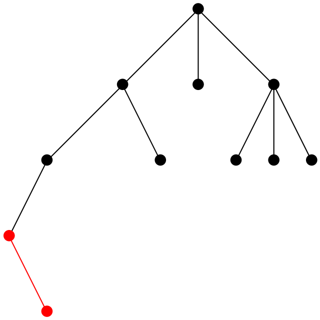

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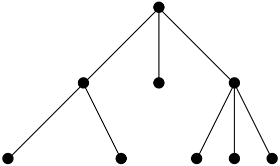

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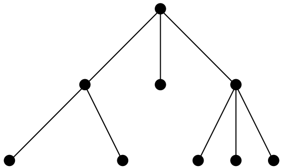
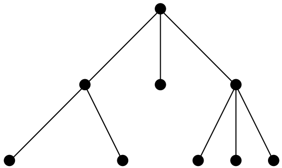


- Every available move remains available if it is not played

0.03 on trees

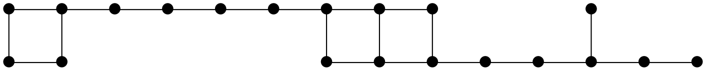


- Every available move remains available if it is not played
- Exception : P_3, in which both available moves are equivalent

0.03 on $2 \times n$ grids

Definition

An *even* $(1-2)$-*grid graph* is a connected induced subgraph of a $2 \times n$ grid where each block of consecutive columns of size 1 has an even size.



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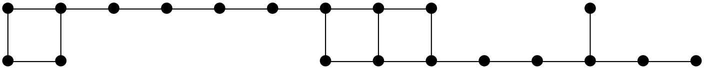

Proposition

From a non-empty even $(1,2)$-grid graph, one can only play to an even $(1,2)$-grid graph.

0.03 on $2 \times n$ grids

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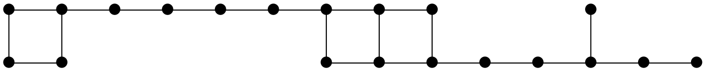


Proposition

From a non-empty even $(1,2)$-grid graph, one can only play to an even $(1,2)$-grid graph.

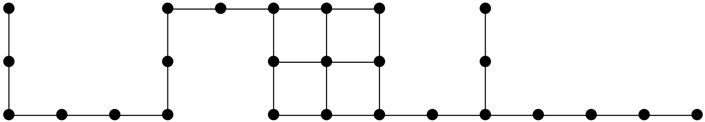
Theorem

At the end of the game on a $2 \times n$ grid, the grid will be empty.

0.03 on $3 \times n$ grids

Définition

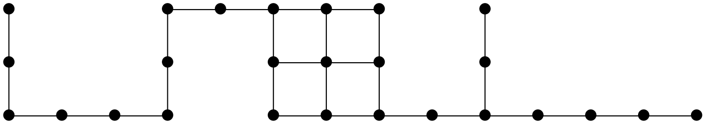
A $(1,3)$-grid graph is a connected induced subgraph of a $3 \times n$ grid, whose columns are of size 1 or 3.



0.03 on $3 \times n$ grids

Définition

A $(1,3)$-grid graph is a connected induced subgraph of a $3 \times n$ grid, whose columns are of size 1 or 3. Moreover, if a column is of size 1, then its vertex is not in the middle row.



0.03 on $3 \times n$ grids

Lemma

One can always play from a non-empty $(1, 3)$-grid to a $(1, 3)$-grid.

0.03 on $3 \times n$ grids

Lemma
One can always play from a non-empty $(1,3)$-grid to a $(1,3)$-grid.

Lemma
For every move on an $(1,3)$-grid of size ≥ 4, there is an answer resulting in an $(1,3)$-grid.

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For every move on an $(1,3)$-grid of size ≥ 4, there is an answer resulting in an $(1,3)$-grid.

Theorem

There is a winning strategy emptying a $3 \times n$ grid for the game 0.03.

0.03 on $3 \times n$ grids

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For every move on an $(1, 3)$-grid of size ≥ 4, there is an answer resulting in an $(1, 3)$-grid.

Theorem

There is a winning strategy emptying a $3 \times n$ grid for the game 0.03.

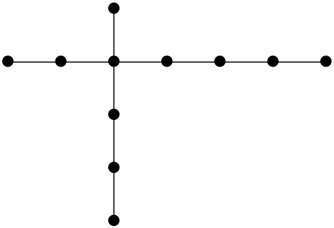
Conjecture

There is a winning strategy emptying an $m \times n$ grid.

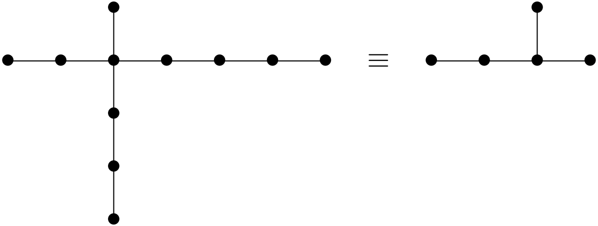
Section 3

0.33 on graphs

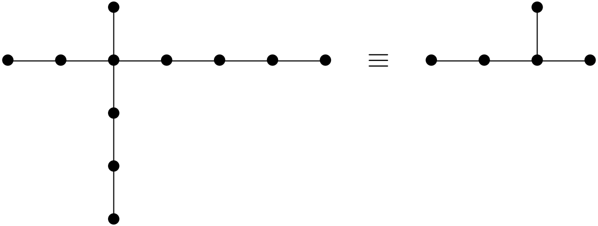
0.33 on subdivided stars



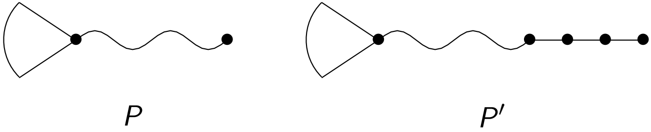
0.33 on subdivided stars



0.33 on subdivided stars

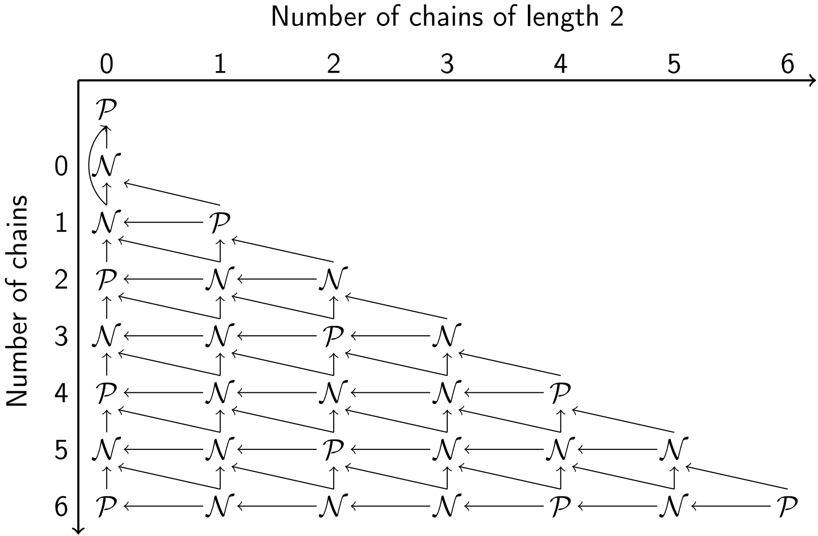


Proof

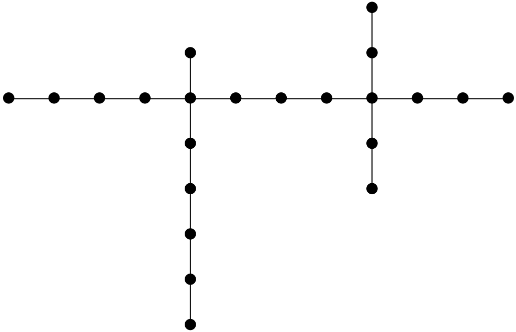


We prove that $P \equiv P'$ by proving that $P + P'$ is a $\mathcal{P}$-position.

0.33 on subdivided stars



0.33 on subdivided bistars



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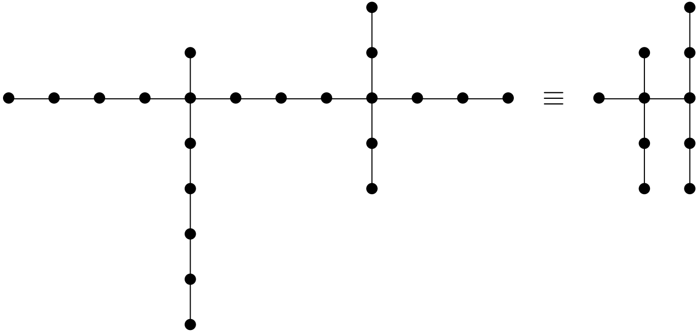
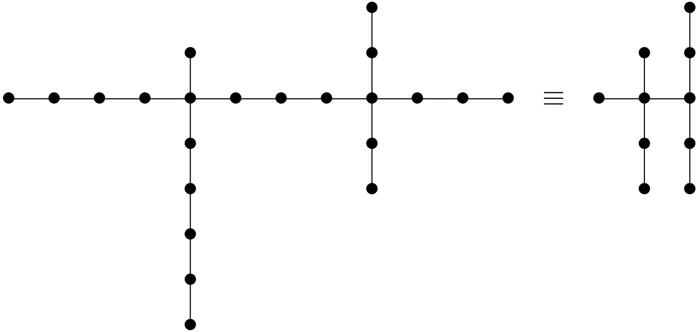


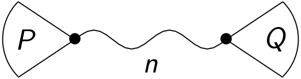
0.33 on subdivided bistars



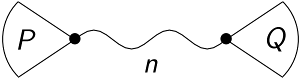
Idea of the proof

One can not play in the middle chain before emptying at least one of the two stars.

Emergence of a pseudo-sum

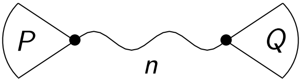


Emergence of a pseudo-sum



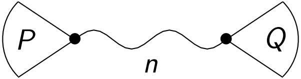
- Can we compute the value of the bistar with the values of P and Q ?

Emergence of a pseudo-sum



- Can we compute the value of the bistar with the values of P and Q ?
- Yes, but it depends on n !

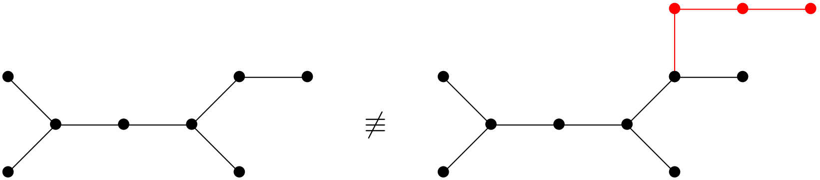
Emergence of a pseudo-sum



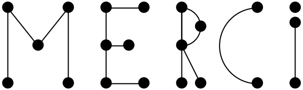
- Can we compute the value of the bistar with the values of P and Q ?
- Yes, but it depends on n ! If $n = 1$, the pseudo-sum is defined as such:

	$\mathcal{C}_0$	$\mathcal{C}_1$	$\mathcal{C}_1^*$	$\mathcal{C}_2$	$\mathcal{C}_2^*$	$\mathcal{C}_2^\square$	$\mathcal{C}_3$	$\mathcal{C}_3^\square$
$\mathcal{C}_0$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
$\mathcal{C}_1$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
$\mathcal{C}_1^*$	$\oplus$	$\oplus$	2	$\oplus$	0	$\oplus$	$\oplus$	$\oplus$
$\mathcal{C}_2$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
$\mathcal{C}_2^*$	$\oplus$	$\oplus$	0	$\oplus$	1	1	$\oplus$	0
$\mathcal{C}_2^\square$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	1	$\oplus$	$\oplus$	$\oplus$
$\mathcal{C}_3$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$
$\mathcal{C}_3^\square$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	1	$\oplus$	$\oplus$	$\oplus$

0.33 on trees ?



The end !



Any questions ?