A strengthening of the Murty-Simon Conjecture on diameter 2 critical graphs

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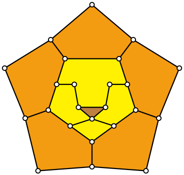

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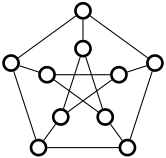

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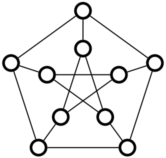

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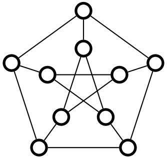
1. It has diameter d;
2. Deleting any edge increases the diameter.



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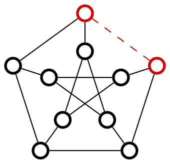

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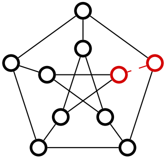

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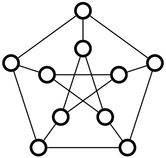

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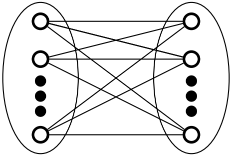
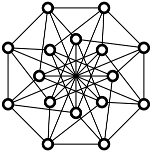


Complete bipartite graphs



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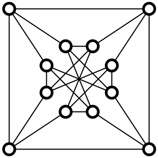

Chvàtal Graph

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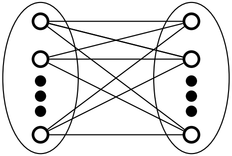

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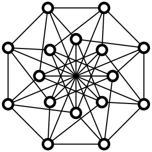

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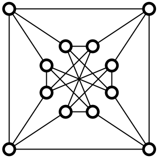

Chvàtal Graph

... and many others!

The Murty-Simon Conjecture

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Theorem (Mantel, 1907)

A triangle-free graph of order n and size m verifies $m \leq \lfloor \frac{n^2}{4} \rfloor$. The extremal graph is $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

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- With a dominating edge (Hanson and Wang, 2003, Haynes *et al.*, 2011, Wang 2012)

The Murty-Simon Conjecture: a linear strengthening?

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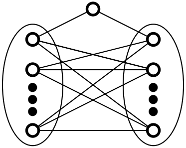
Claim (Füredi, 1992)

A non-bipartite D2C graph of order $n \geq n_0$ and size m verifies $m \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 1 \approx \lfloor \frac{n^2}{4} - \frac{n}{2} \rfloor$. The extremal graph is obtained by subdividing an edge of $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.



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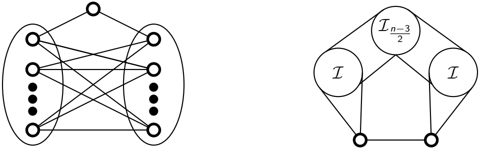

Theorem (Balbuena *et al.*, 2015)

A triangle-free non-bipartite D2C graph of order n and size m verifies $m \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 1$. The extremal graphs are some inflations of C_5.

Strengthening the Murty-Simon Conjecture

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A non-bipartite D2C graph of order $n > 6$ and size m verifies $m \leq \lfloor \frac{(n-1)^2}{4} \rfloor + 1$. If $n \geq 10$, the extremal graphs are some inflations of C_5.

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Let c be a positive integer, then there is a rank n_0 such that any non-bipartite D2C graph of order $n \geq n_0$ and size m verifies $m < \lfloor \frac{n^2}{4} \rfloor - c$.

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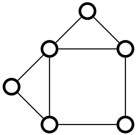
$H_5 =$ 

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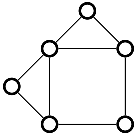
$H_5 =$ 

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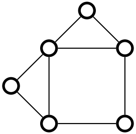
$H_5 =$ 

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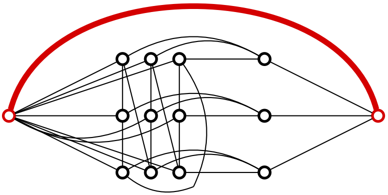
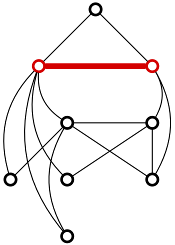
$m \geq \lfloor \frac{n^2}{4} \rfloor - c = 9 - c$ if $c \geq 1$

Our main result

Theorem (D., Foucaud, Hansberg, 2018)

Let G be a non-bipartite D2C graph with a dominating edge of order n and size m. If $G \neq H_5$ then $m < \lfloor \frac{n^2}{4} \rfloor - 1$.





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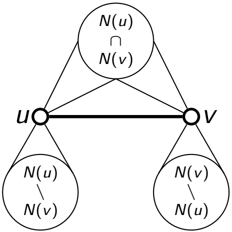

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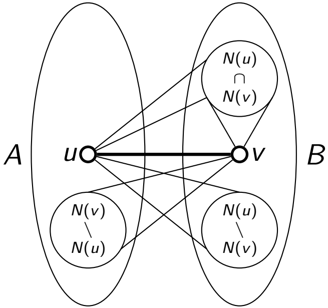

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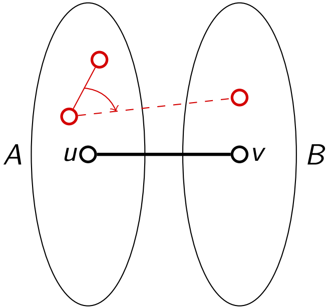

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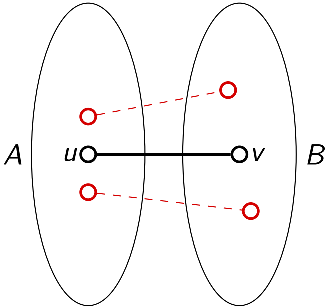
3. Find two non-assigned non-edges between A and B.



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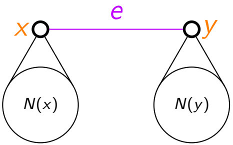

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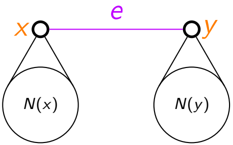

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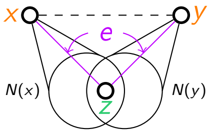

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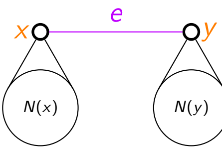

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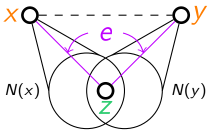

$\Rightarrow$ In a D2C graph, every edge is critical for some pair of vertices

Assign internal edges to external non-edges

Lemma

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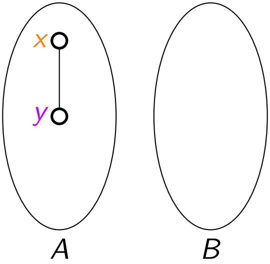

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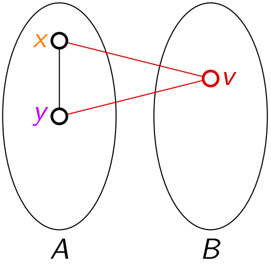
Let xy be an edge in A. It is not critical for x and y since they have v as a common neighbour.



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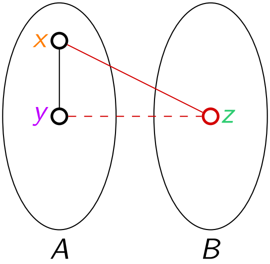

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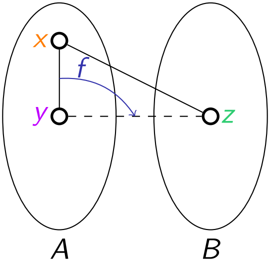

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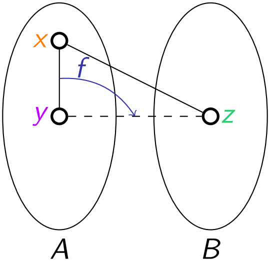

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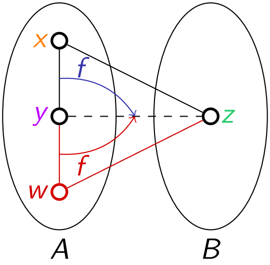

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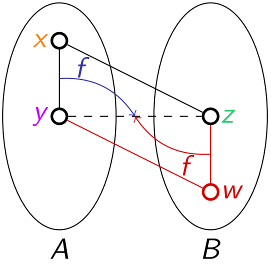

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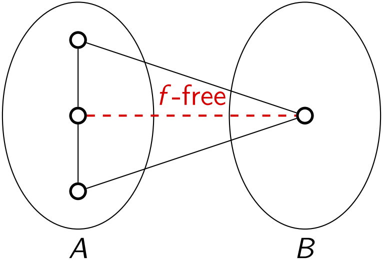

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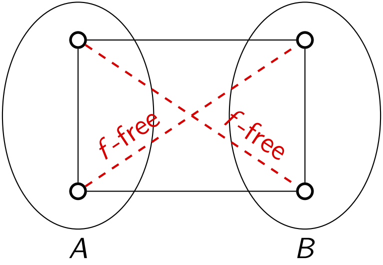

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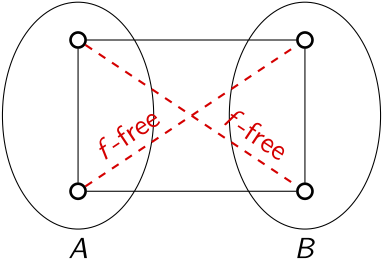
There are $\frac{n^2 - \left| |A| - |B| \right|^2}{4} - \mathrm{free}(f)$ edges in the graph.



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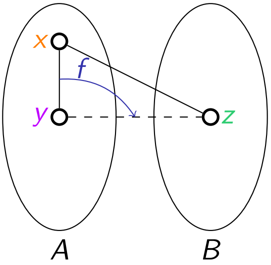
Lemma

1. $N(u) \cap N(v) = \emptyset$
2. uv is critical for u and v only
3. There is at least one edge in A or B

Defining an orientation of the internal edges

Definition

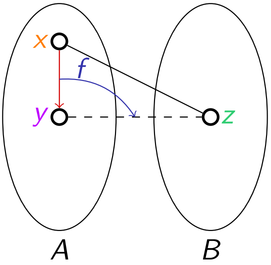
If $f(xy) = \overline{yz}$,



Defining an orientation of the internal edges

Definition

If $f(xy) = \overline{yz}$, we orient xy from x to y. This defines an f-orientation.



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There is no directed cycle.

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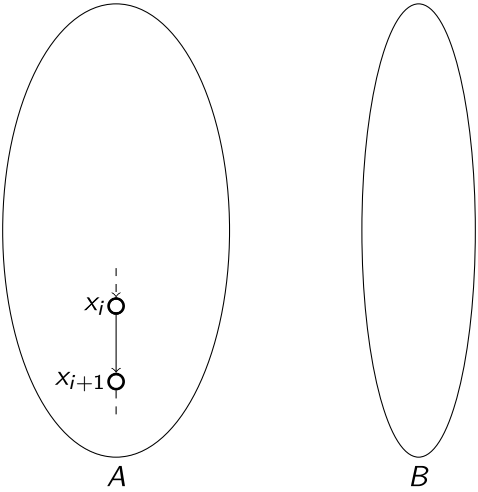

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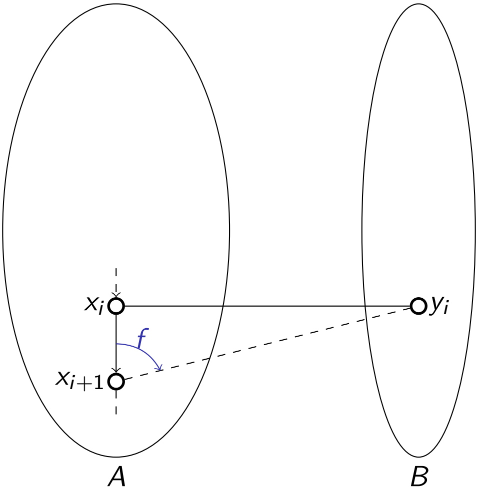

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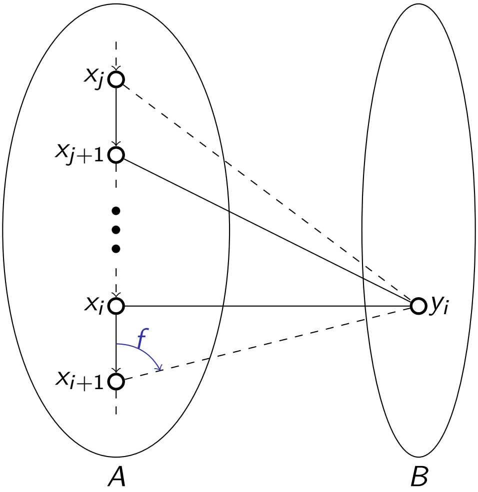

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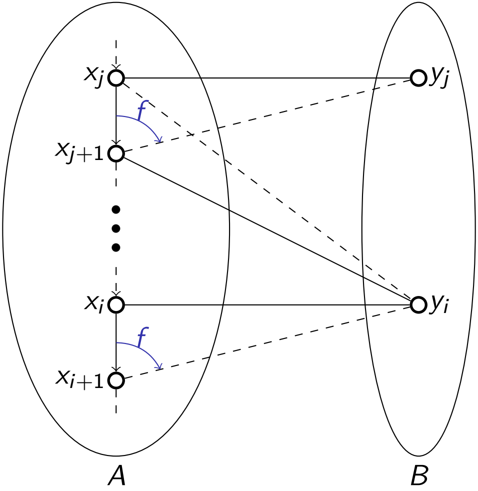

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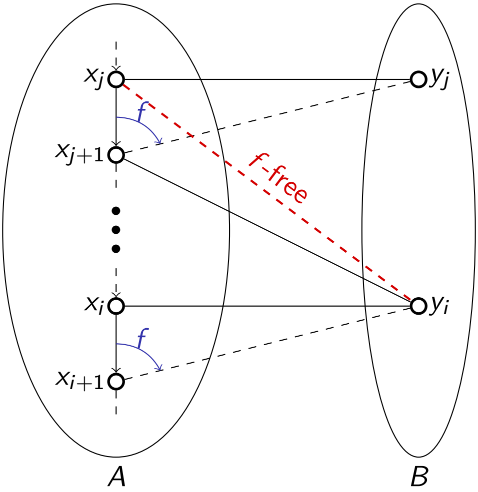

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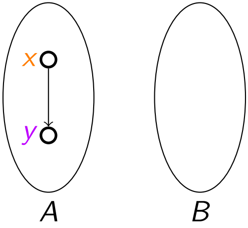

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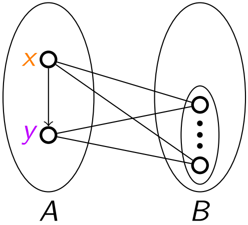

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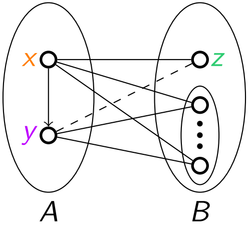

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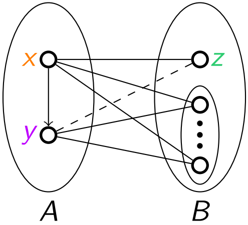

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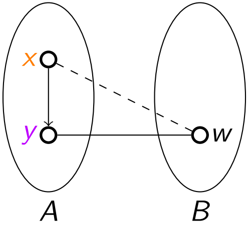

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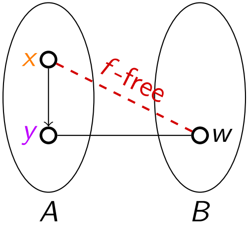

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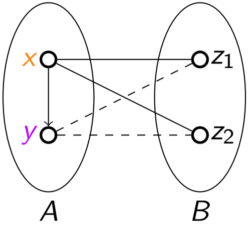

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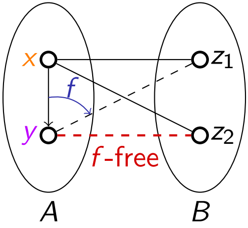

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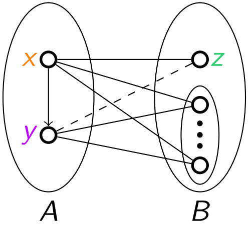

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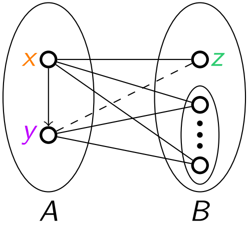

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Fact to contradict: there is ≤ 1 f-free non-edge

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1. Only one nontrivial component in A and B

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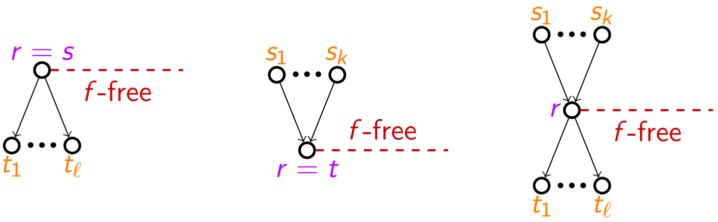

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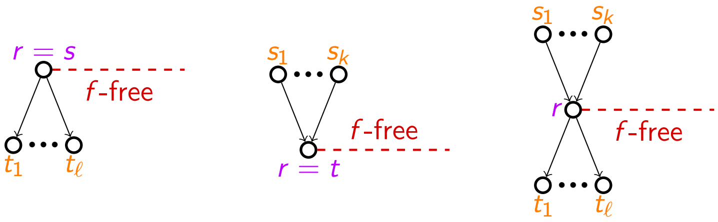

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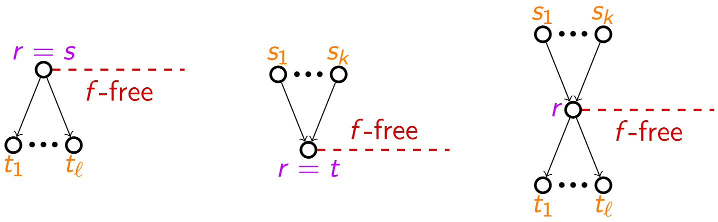

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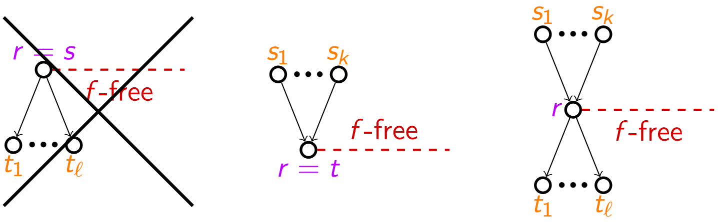

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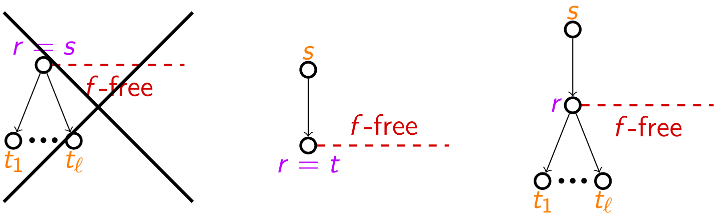

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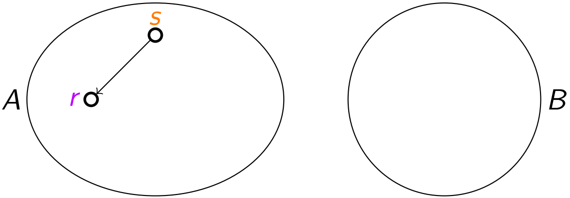
The last steps



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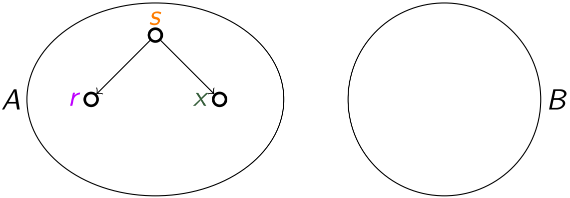

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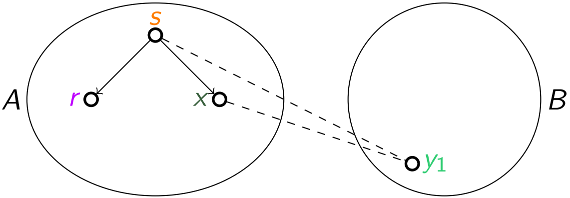

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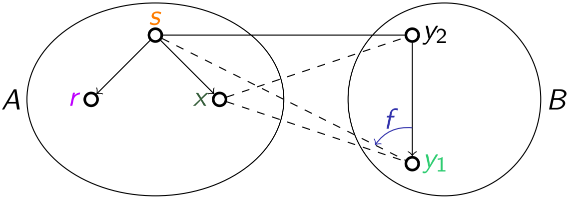

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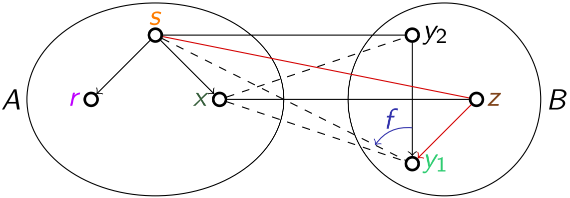

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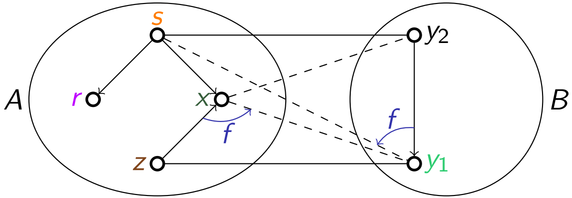

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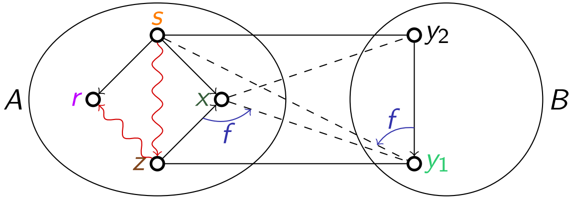

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