Neighbour sum-distinguishing edge colorings with local constraints

Antoine Dailly[1], Éric Duchêne[2],
Aline Parreau[2], Elżbieta Sidorowicz[3]

ICGT 2022

[1] G-SCOP, Grenoble
[2] LIRIS, Lyon
[3] University of Zielona Góra, Pologne

Distinguishing colorings

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Examples

$\sigma_\omega(u)$	Global	Local
$\bigcup_{v \in N(u)} \omega(uv)$	[Harary & Plantholt, 1985]	[Györi *et al.*, 2008]
$\sum_{v \in N(u)} \omega(uv)$	[Chartrand *et al.*, 1988]	[Karoński *et al.*, 2004]
$\prod_{v \in N(u)} \omega(uv)$	Undefined	[Skowronek-Kaziów, 2008]

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$+\ \omega$ proper	[Burris & Schelp, 1997]	[Zhang *et al.*, 2002]
$\sum_{v \in N(u)} \omega(uv)$	[Chartrand *et al.*, 1988]	[Karoński *et al.*, 2004]
$+\ \omega$ proper	[Lo, 1985]	[Flandrin *et al.*, 2013]
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$\rightarrow$ We are focusing on local sum-distinguishing edge colorings

Sum-distinguishing edge coloring

Definition

Let ω be a k-edge coloring of G. We define the vertex-coloring $\sigma_\omega : \sigma_\omega(u) = \sum_{v \in N(u)} \omega(uv)$.
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1-2-3 Conjecture (Karoński, Luczak, Thomason, 2004)

If no restriction on ω, then at most 3 colors are enough.

Conjecture (Flandrin, Marczyk, Przybylo, Sacle, Woźniak, 2013)

If ω is proper and $G \neq C_5$, then $k \leq \Delta(G) + 2$.

State of the art

1-2-3 Conjecture

- Best general bound: 5 [Kalkowski, Karoński, Pfender, 2011]
- Holds for 3-colorable graphs [Karoński *et al.*, 2004], 2 are enough for trees [Chang *et al.*, 2011]
- Holds for graphs large enough and very dense ($\delta(G) > 0.99985n$) [Zhong, 2019] ($\delta(G) \geq C \log(\Delta(G))$) [Przybyło, 2020+]
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Proper variant

- True for trees, K_n, $K_{n,n}$ [Flandrin *et al.*, 2013]
- Bound of $\lceil \frac{10\Delta(G)+2}{3} \rceil$ [Wang & Yan, 2014]
- Bound of 6 for subcubic graphs [Huo *et al.* and Yu *et al.*, 2017]

d-relaxed sum-distinguishing edge coloring

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Definition (D., Duchêne, Parreau, Sidorowicz, 2022)

A sum-distinguishing k-edge coloring is *d-relaxed* if every vertex is incident with **at most** d edges of the same color.

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For every connected $G \notin \{K_2, C_5\}$, $\chi'^{d}_{\sum}(G) \leq \left\lceil \frac{\Delta(G)}{d} \right\rceil + 2$.

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- Trees: $\chi'^{d}_{\sum}(T) =$
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- Complete graphs:
  - $d \in \{\lceil \frac{n-1}{2} \rceil, \dots, n-2\} \Rightarrow \chi'^d_\sum(K_n) \leq 4$
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- Subcubic graphs: $\chi'^2_\sum(G) \leq 4$ and every vertex of degree 2 has incident edges of different colors

Complete graphs, $d = 2$

Theorem (D., Duchêne, Parreau, Sidorowicz, 2022)

Let $n \geq 4$. Then: $\chi'^{2}_{\sum}(K_n) = \begin{cases} \lceil \frac{n-1}{2} \rceil + 1 & \text{if } n \not\equiv 3 \bmod 4 \\ \lceil \frac{n-1}{2} \rceil + 2 & \text{if } n \equiv 3 \bmod 4 \end{cases}$

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Proof in two steps

1. Constructing such a 2-relaxed distinguishing coloring
2. Necessary to use this many colors

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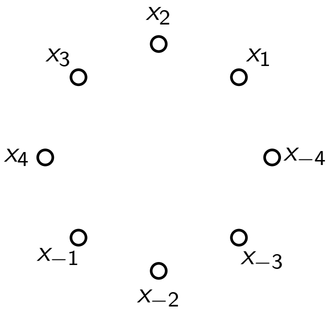
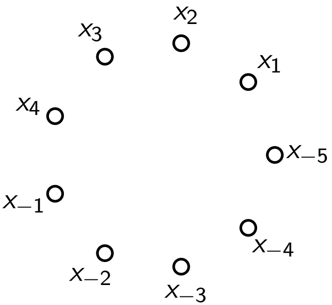
   1.1 Construction of the 2-relaxed coloring
   1.2 Recoloring to make it distinguishing
2. Necessary to use this many colors

Complete graphs, $d = 2$: initial construction





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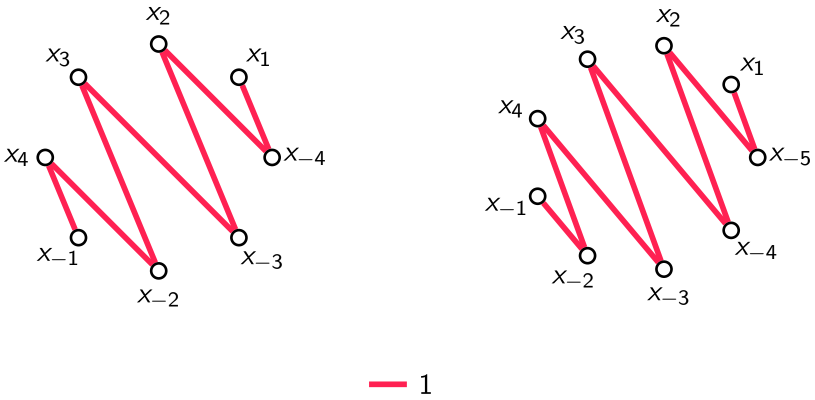

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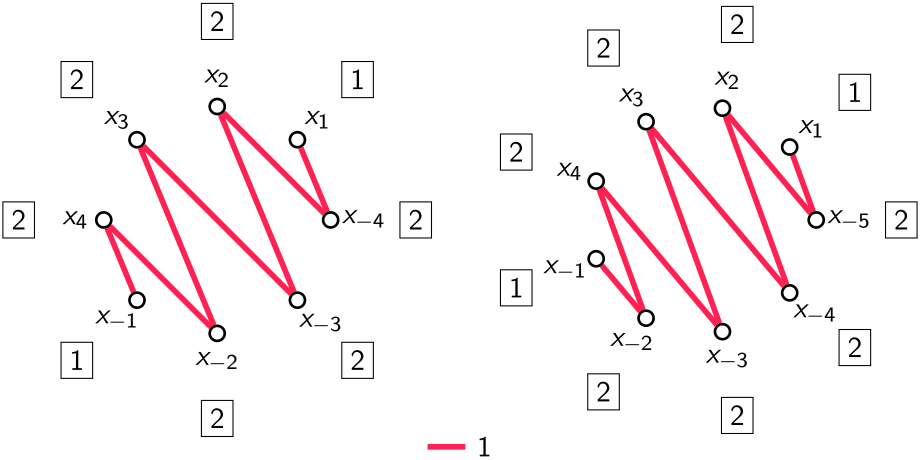

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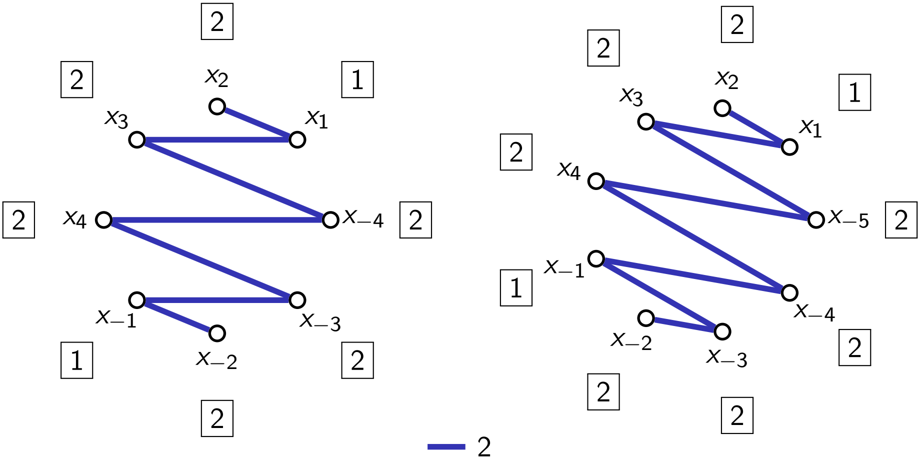

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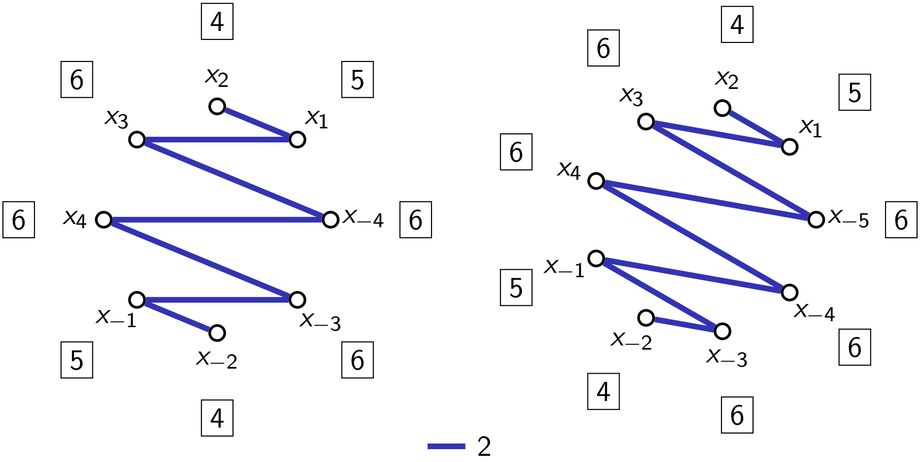

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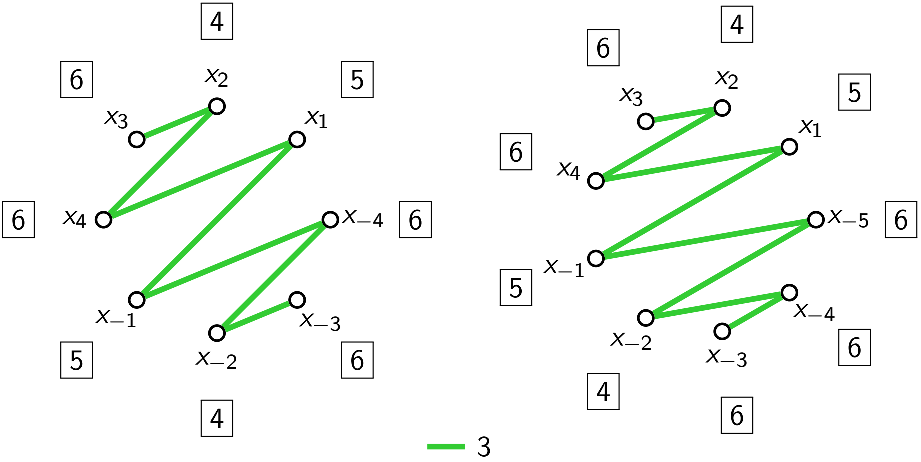

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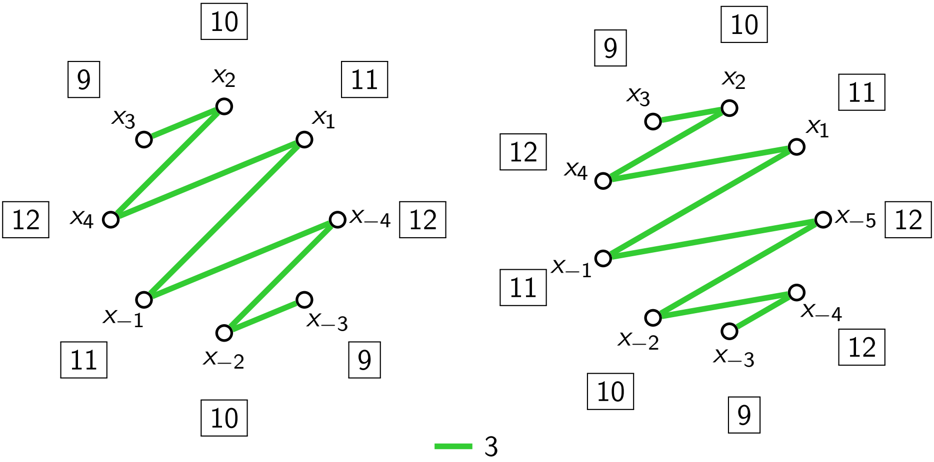

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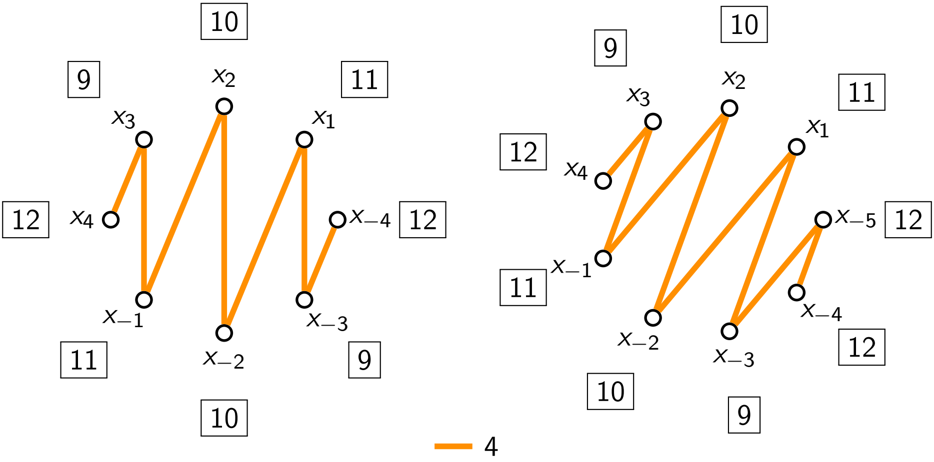

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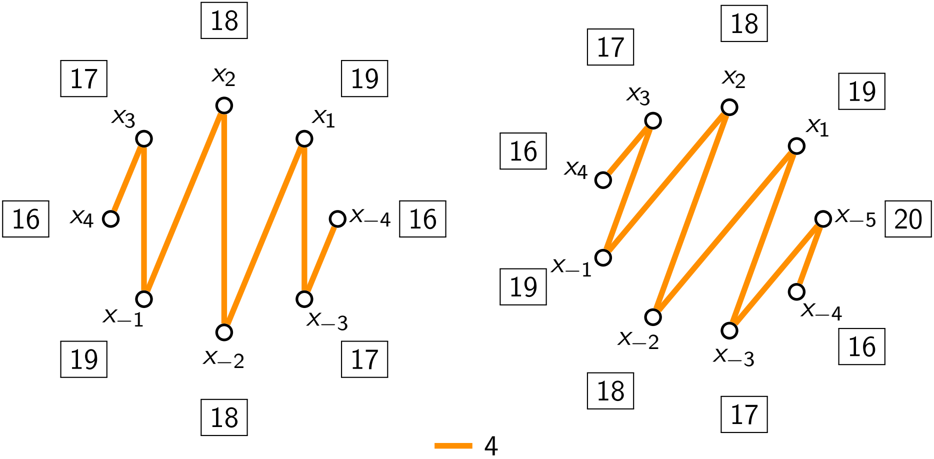

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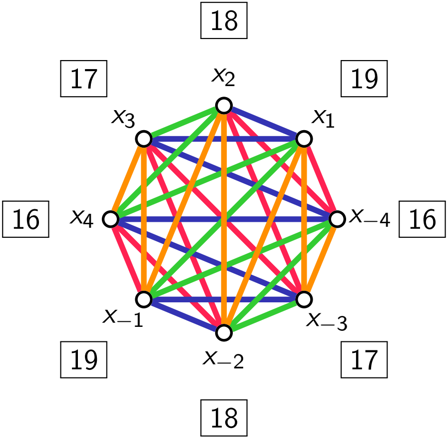

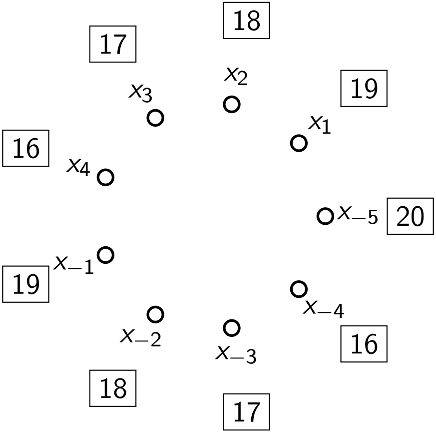

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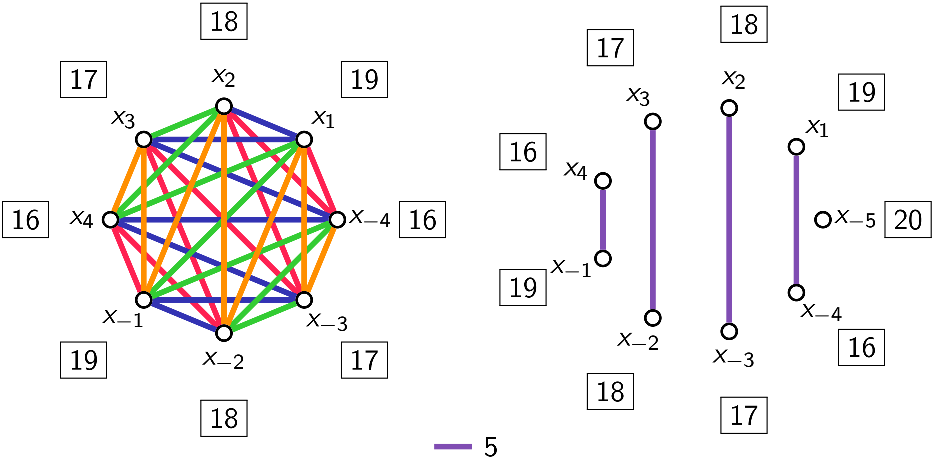

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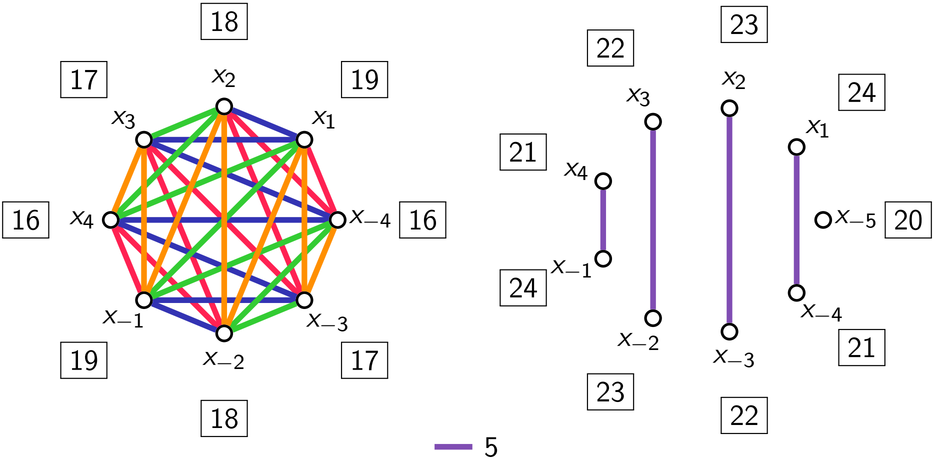

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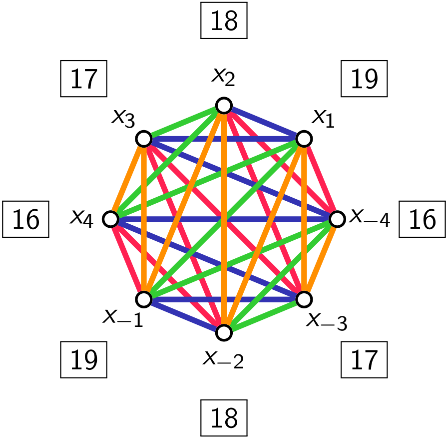

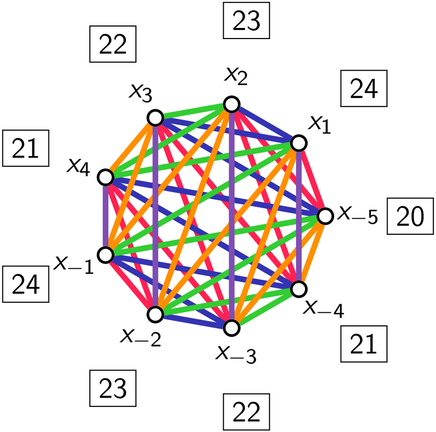

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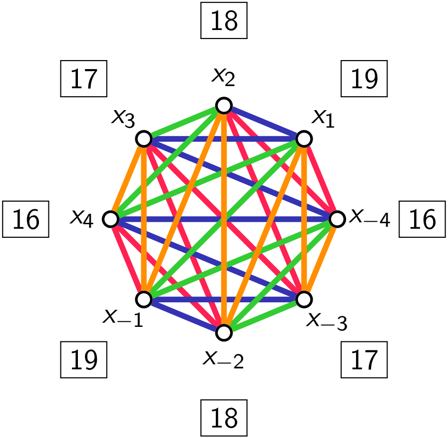

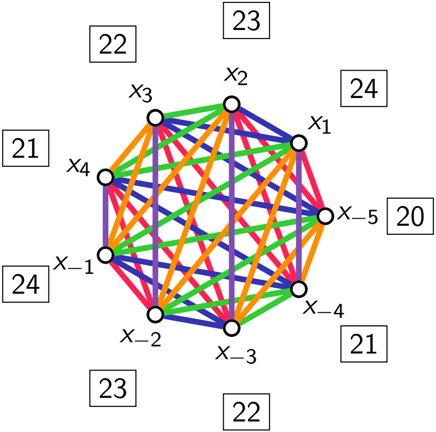

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- 2-relaxed coloring of K_n

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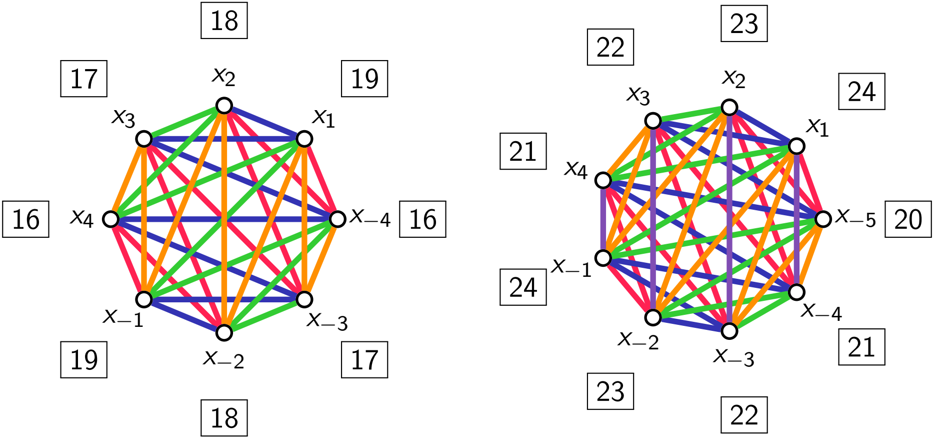

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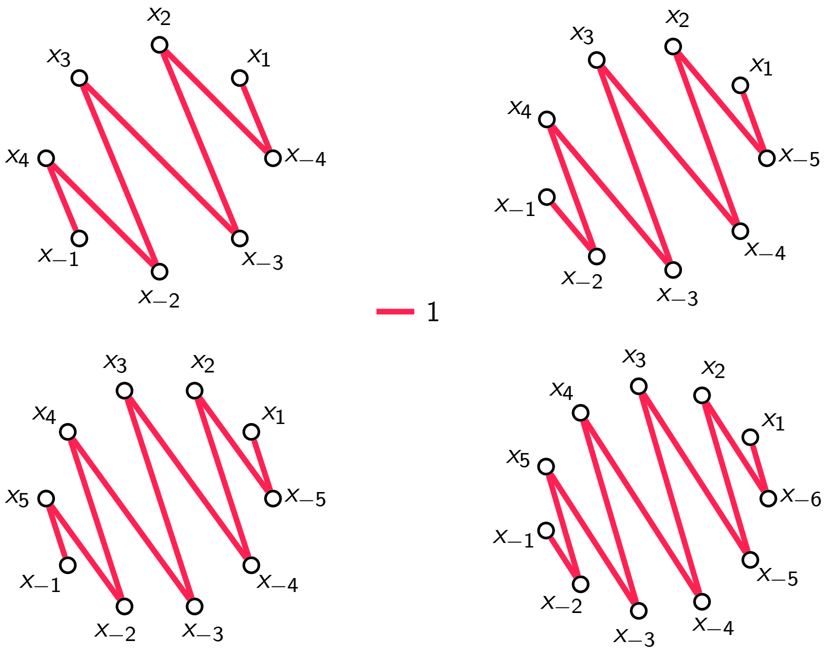
- 2-relaxed coloring of K_n
- x_i and x_{-i} are not distinguished

Complete graphs, $d = 2$: recoloring



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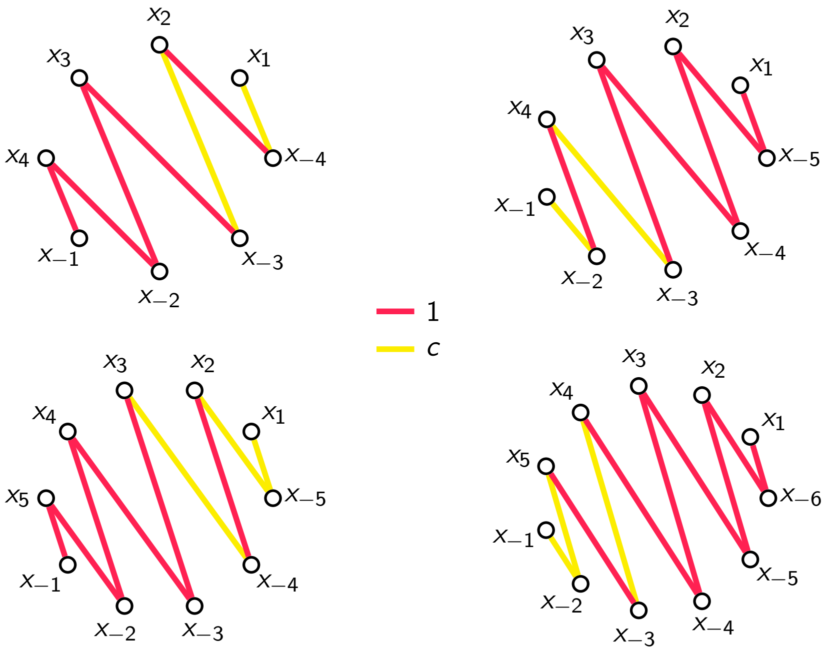

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Theorem (D., Duchêne, Parreau, Sidorowicz, 2022)

Every subcubic graph $G \notin \{K_2, C_5\}$ admits a 2-relaxed sum-distinguishing 4-edge coloring such that every degree 2 vertex is incident with two edges of different colors.

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Four cases depending on the girth, with more subcases...

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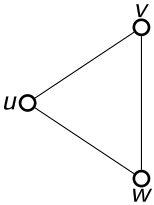

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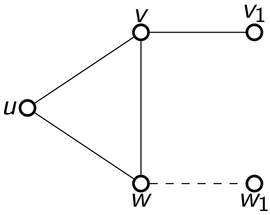

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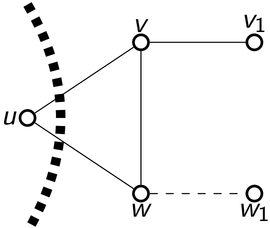

Coloring
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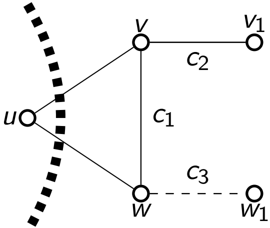

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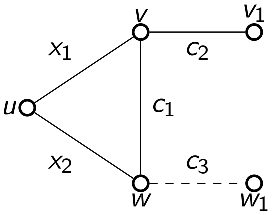

Extension
to G

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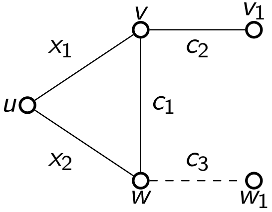

Extension to G

Whether w_1 exists or not: at most 2 forbidden values for x_1 and x_2 to distinguish u from v and w.
Example: w_1 does not exist $\Rightarrow x_1, x_2 \neq c_1$ and $x_2 \neq c_1 + c_2$.

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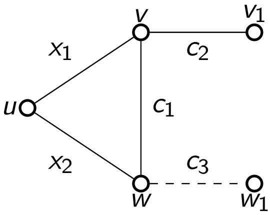

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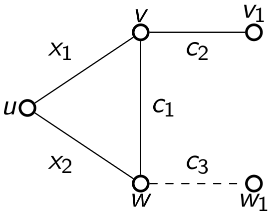

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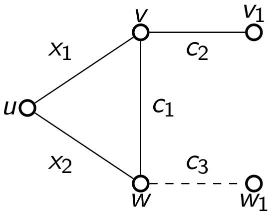

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Let $P(x_1, x_2) = (x_1 - x_2)(x_1 + c_2 - x_2 - c_3)$.

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Combinatorial Nullstellensatz (Alon, 1999)

Let $P(x_1, \ldots, x_n)$ be a polynomial over a field F and $x_1^{k_1} \ldots x_n^{k_n}$ be a monomial of nonzero coefficient and maximal degree in P. For each $S_1, \ldots, S_n \subseteq F$ such that $|S_i| > k_i$, there are $a_1 \in S_1, \ldots, a_n \in S_n$ such that $P(a_1, \ldots, a_n) \neq 0$.

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The monomial $x_1 x_2$ has coefficient 2 and maximal degree in P, and $|S_1|, |S_2| > 1 \Rightarrow$ We can extend the coloring

Beyond subcubic graphs

Theorem (D., Duchêne, Parreau, Sidorowicz, 2022)

Every subcubic graph $G \notin \{K_2, C_5\}$ admits a 2-relaxed sum-distinguishing 4-edge coloring such that every degree 2 vertex is incident with two edges of different colors.

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Every graph G with $\Delta(G) \leq 4$ (resp. 5) admits a sum-distinguishing 6-edge coloring (resp. 7-edge coloring) such that every vertex of degree at least 2 is incident with at least two edges of different colors.

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Stronger local constraint, but weaker bound!

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Proof idea

Adaptation of an algorithm by Kalkowski (2009+).

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1. Define a vertex ordering with specific properties
2. Every edge receives color 4
3. Consider each vertex in the order
   3.1 The only edges that can be modified are between the vertex, its predecessors, and its first successor
   3.2 Ensure that the coloring is sum-distinguishing and that every vertex of degree at least 6 is incident with a non-monochromatic set of edges

$\rightarrow$ Several cases are considered

Beyond subcubic graphs

Corollary

Let G be a graph. We have:

- $\Delta(G) \leq 3 \Rightarrow \chi'^{\Delta(G)-1}_{\sum}(G) \leq 4$
- $\Delta(G) \leq 4 \Rightarrow \chi'^{\Delta(G)-1}_{\sum}(G) \leq 6$
- $\Delta(G) \leq 5 \Rightarrow \chi'^{\Delta(G)-1}_{\sum}(G) \leq 7$

Corollary

Every graph G verifies $\chi'^{\Delta(G)-1}_{\sum}(G) \leq 7$.

Conclusion

Conjecture (D., Duchêne, Parreau, Sidorowicz, 2022)

For every connected $G \notin \{K_2, C_5\}$, $\chi'^{d}_{\sum}(G) \leq \left\lceil \frac{\Delta(G)}{d} \right\rceil + 2$.

$\rightarrow$ Generalization of the 1-2-3 Conjecture and its proper variant

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Open questions

- Complete graphs: $d \in \{3, \ldots, \lceil \frac{n-1}{2} \rceil - 1\}$, exact value for $d \in \{\lceil \frac{n-1}{2} \rceil, \ldots, n-2\}$
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