Path Covers of Temporal Graphs: When is Dilworth dynamic?

Dibyayan Chakraborty[1], Antoine Dailly[2], Florent Foucaud[2], Ralf Klasing[3]

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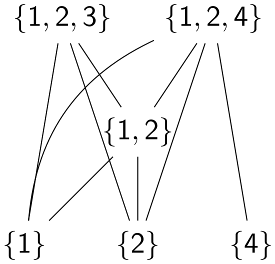
[1] School of Computing, University of Leeds
[2] LIMOS, Clermont-Ferrand
[3] LaBRI, Bordeaux

ANR GRALMECO and TEMPOGRAL

Introduction: Dilworth's theorem



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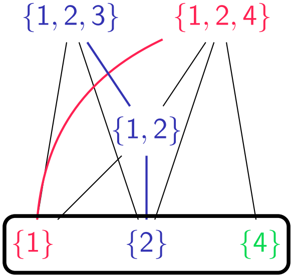
Theorem [Dilworth, 1950]

The minimum size of a chain partition of a finite poset is equal to the maximum size of an antichain of this poset.



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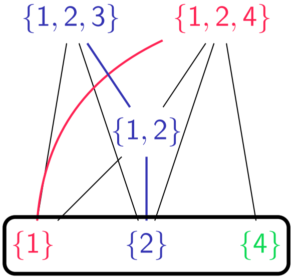
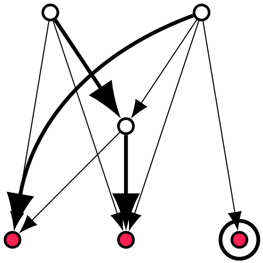
The minimum size of a path partition of a transitive DAG is equal to the maximum size of an antichain of this DAG.

Restated for graphs...





Introduction: Dilworth's theorem

Theorem [Dilworth, 1950]

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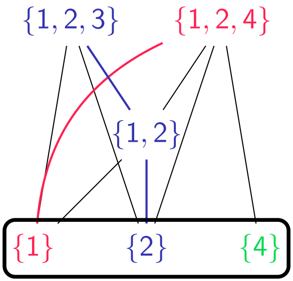

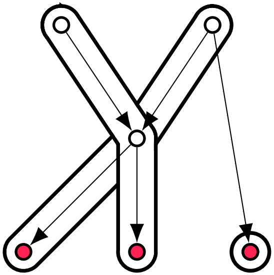

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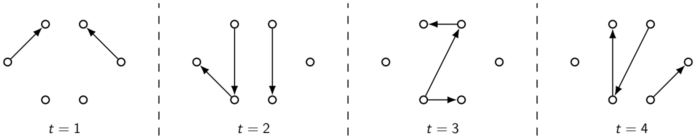
- Many improvements since then, now quasi-linear [Caceres, ICALP 2023]
- NP-hard on general graphs

Introduction: temporal (di)graphs

$\mathcal{D} = (V, A_1, A_2, \ldots, A_k)$ [Ferreira & Viennot, 2002]



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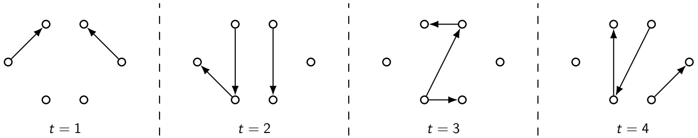
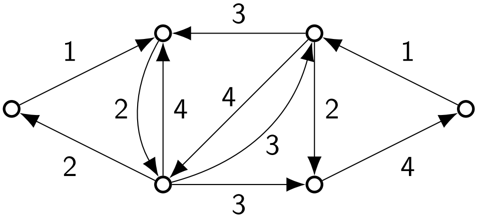


$\mathcal{D} = (V, A, \lambda)$ [Kempe, Kleinberg & Kumar, 2000]



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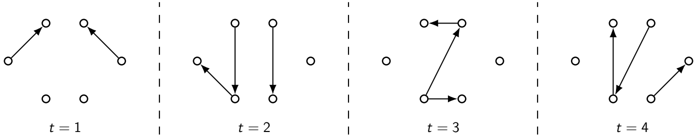

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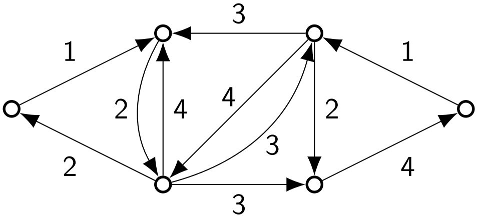
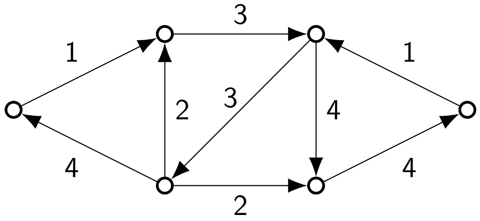


Many results and applications in distributed algorithms, dynamic networks (transportation, social, biological...). More recently, gain of interest from the graph algorithms community.

A few definitions for this talk

- A temporal DAG (resp. tree...) is a temporal (di)graph whose underlying (di)graph is a DAG (resp. tree...).



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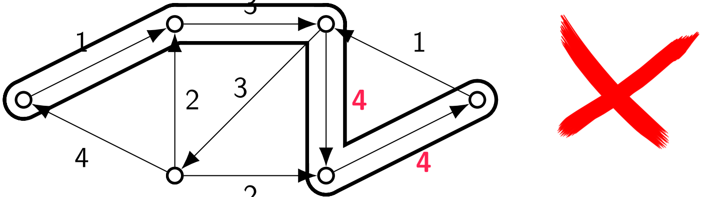

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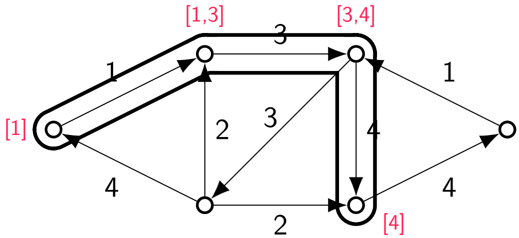

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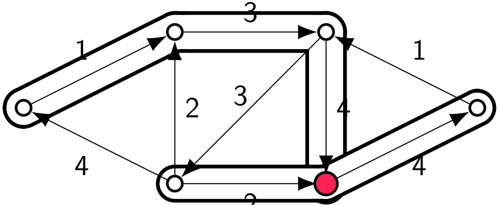

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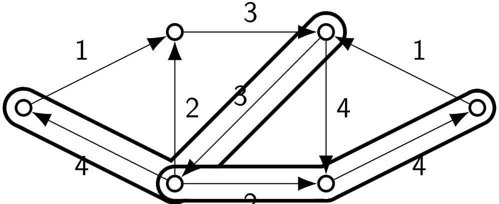

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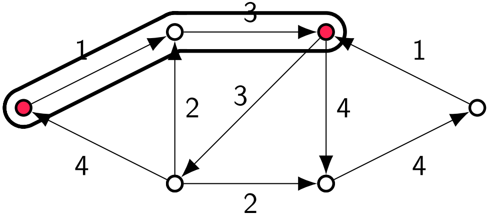

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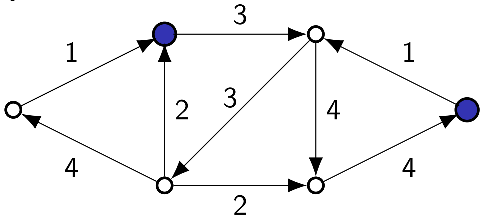
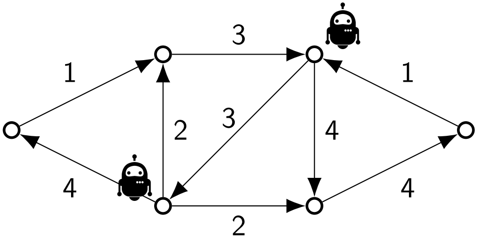
- Two vertices are temporally connected if there is a temporal path between them.
- A temporal antichain is a set of vertices who are pairwise not temporally connected.



Our incentive for temporally disjoint paths

History

- Several papers on paths and journeys in temporal graphs
- Temporally disjoint paths are a good model for dynamic MULTI AGENT PATH FINDING [Stern *et al.*, 2019]



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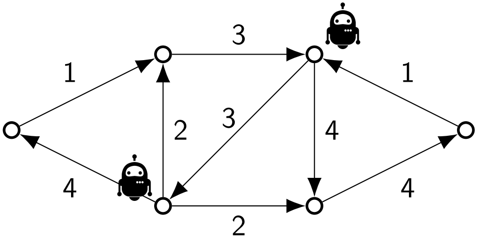

- TEMPORALLY DISJOINT WALKS: W[1]-hard and XP (number of walks) [Klobas *et al.*, IJCAI 2021]
- TEMPORALLY DISJOINT PATHS: NP-hard and W[1]-hard (number of vertices) on temporal stars [Kunz, Molter & Zehavi, IJCAI 2023]

A temporal Dilworth's theorem?

Dilworth property

In a (transitive) DAG, the minimum size of a path partition/cover is equal to the maximum size of a antichain.

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Two problems:

Temporal Path Cover (TPC)

Temporal Path Partition/Temporally Disjoint Path Cover (TD-PC)

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Temporal Path Cover (TPC)

Temporal Path Partition/Temporally Disjoint Path Cover (TD-PC)

Two questions:

Which temporal DAGs have the Dilworth property?

$\Rightarrow$ **Combinatorial** aspect

What is the complexity of those problems?

$\Rightarrow$ **Algorithmic** aspect

Our results

Temporal class	TPC	TD-PC
Oriented paths	$\mathcal{O}(\ell n)$	$\mathcal{O}(\ell n)$
Rooted trees	$\mathcal{O}(\ell n^2)$	$\mathcal{O}(\ell n^2)$
Oriented trees	$\mathcal{O}(\ell n^2 + n^3)$	NP-hard
DAGs*	NP-hard	NP-hard
Digraphs	XP (tw and $t_{\max}$) $n^{\mathcal{O}(\mathrm{tw}^2\, t_{\max} \log(\mathrm{tw}\, t_{\max}))}$	FPT (tw and $t_{\max}$) $2^{\mathcal{O}(\mathrm{tw}^2\, t_{\max} \log(\mathrm{tw}\, t_{\max}))} n$

* planar, subcubic, bipartite, girth 10, $\ell = 1$, $t_{\max} = 2$

$$n = \text{number of vertices}$$
$$\ell = \text{number of (unsorted) time labels per arc}$$
$$t_{\max} = \text{total number of time-steps}$$

For those specific classes, polynomial-time $\Leftrightarrow$ Dilworth property.

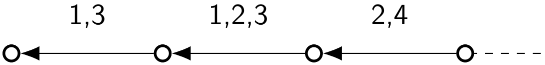
Temporal lines

Theorem [CDFK, 2024+]

Temporal oriented lines have the Dilworth property, and we can solve TPC and TD-PC in time $\mathcal{O}(\ell n)$.

Algorithm

Take a maximum-length temporal path containing a leaf.



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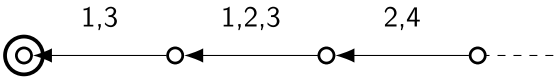

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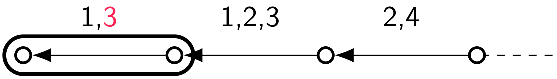

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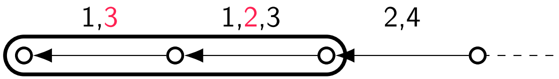

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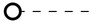

Iterate. The successive leaves are a temporal antichain!

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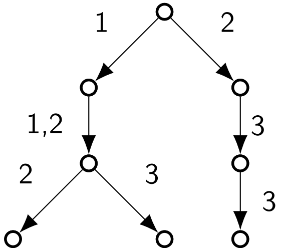

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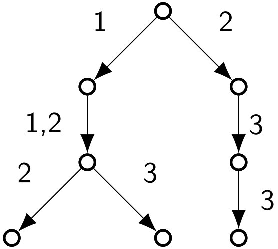

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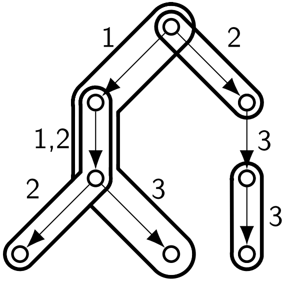

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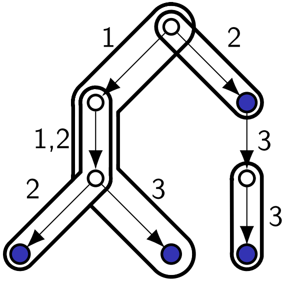

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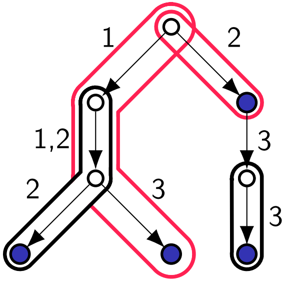

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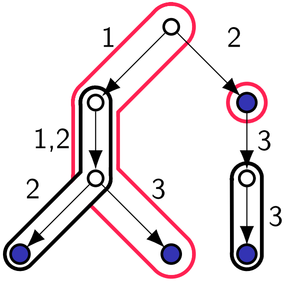

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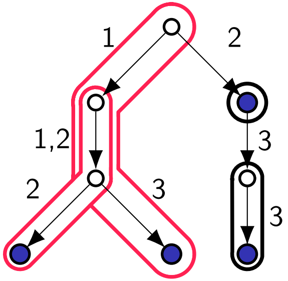

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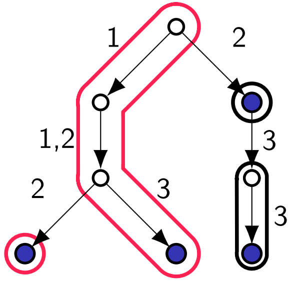

Path Cover of temporal oriented trees (1) Back to static

Theorem [CDFK, 2024+]

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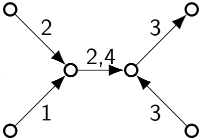
$\mathcal{T} =$ 

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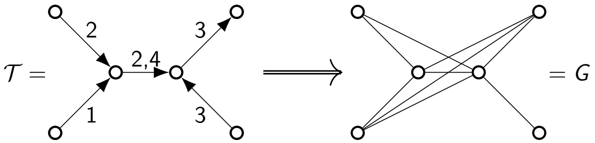

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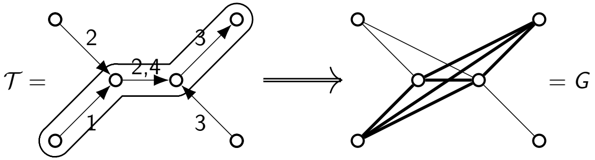

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Clique in G $\Leftrightarrow$ Temporal Path in $\mathcal{T}$.

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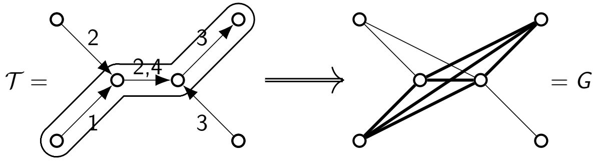

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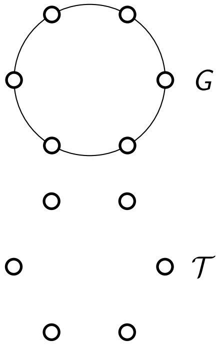

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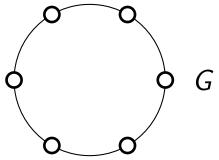

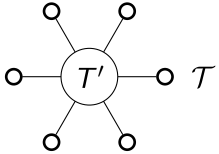

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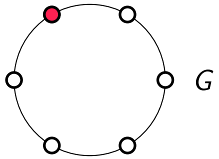

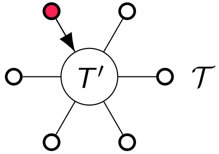

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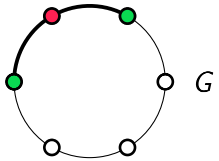

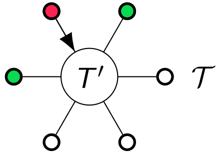

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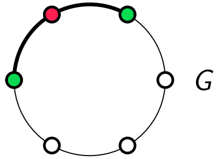

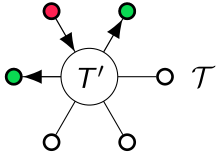

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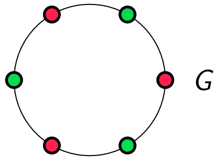

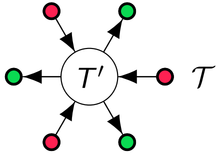

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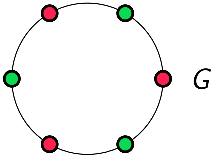

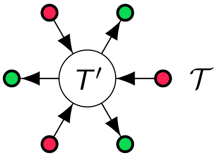

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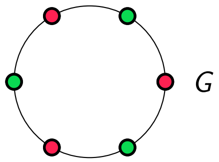
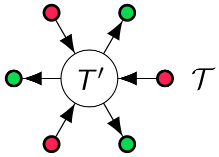
No even hole either (using Helly property and vertex-intersection of temporal paths).





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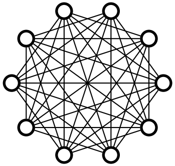
G



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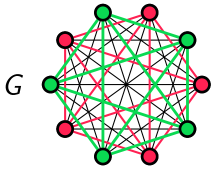

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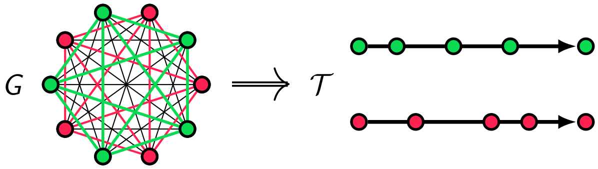

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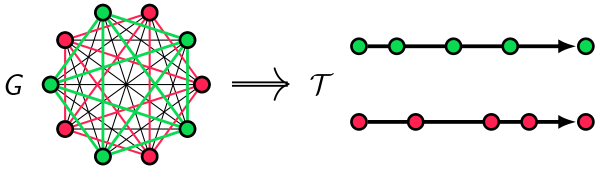

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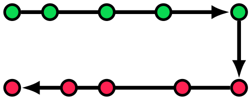

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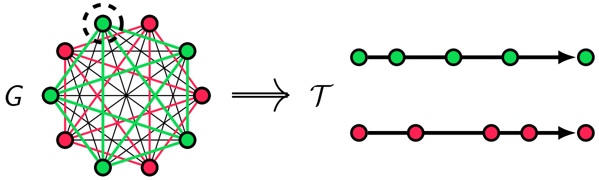

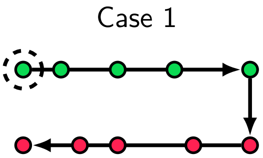

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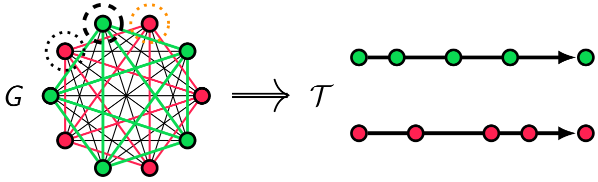

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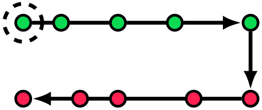

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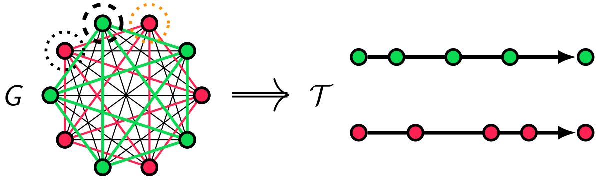

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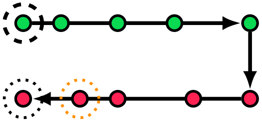

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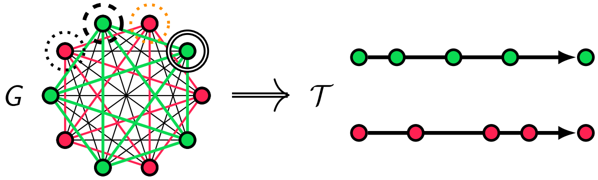

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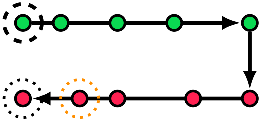

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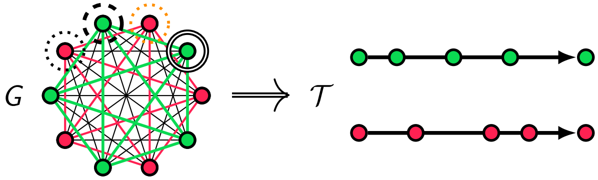

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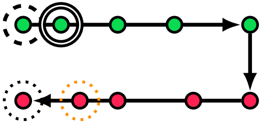

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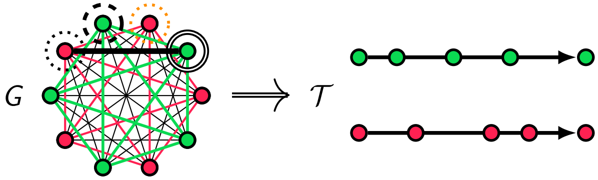

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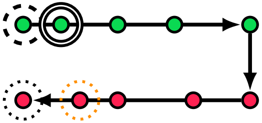

$\Rightarrow$ The temporal path of this edge does not exist in $\mathcal{T}$.

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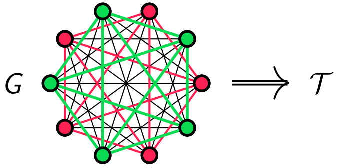

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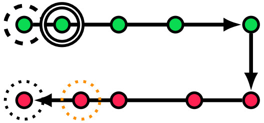

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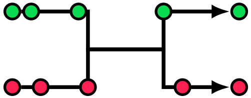

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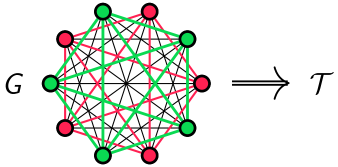

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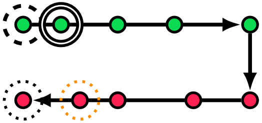

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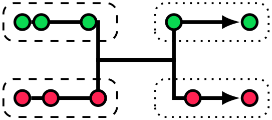

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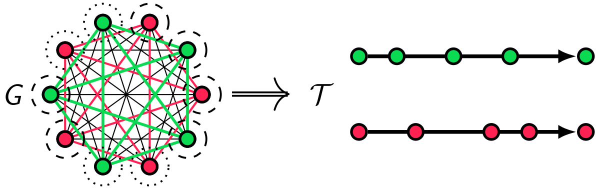

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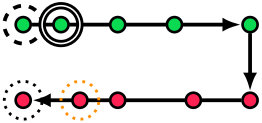

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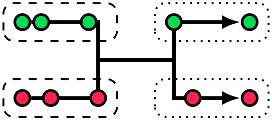

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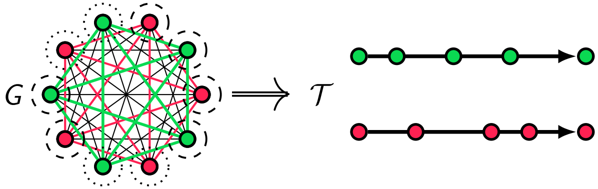

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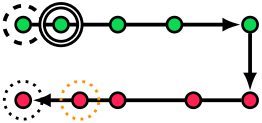

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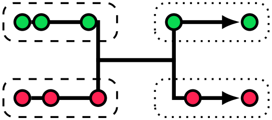

⇒ Each has to be complete bipartite in G, only possible if order $\leq$ 7, which we then manage.

Path Cover of temporal oriented trees (4) Conclusion

Lemmas

The connectivity graph is (hole,antihole)-free

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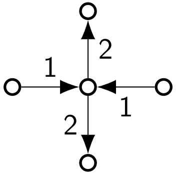

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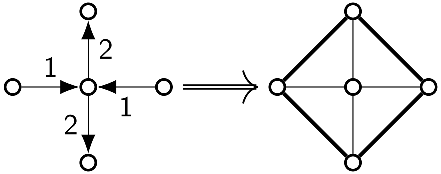

Path Partition of temporal oriented trees: NP-hardness

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TD-PC is NP-hard on temporal oriented trees.

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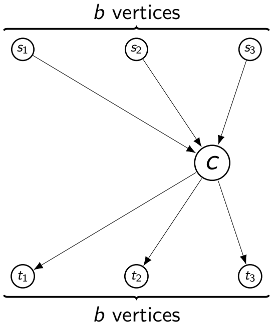

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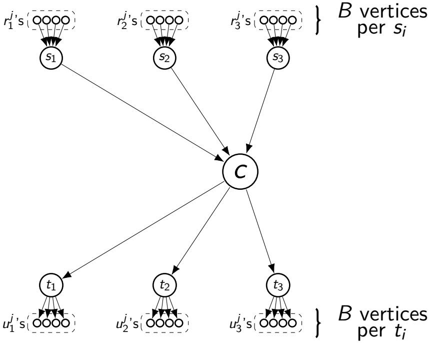

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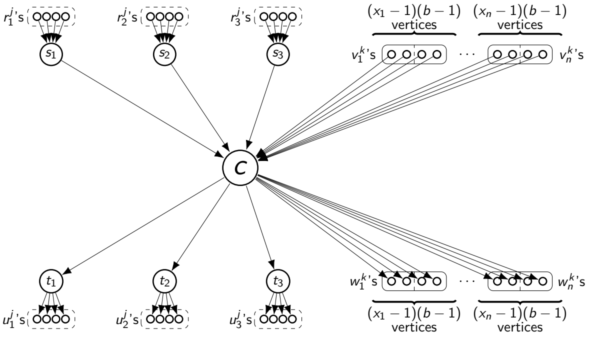

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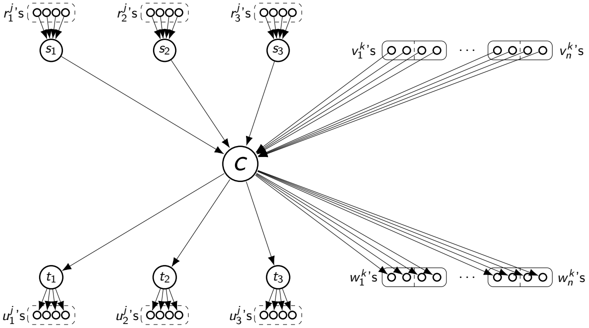

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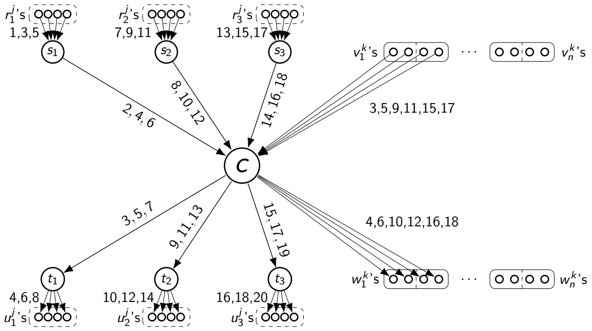

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Here, $x_1 = 3$

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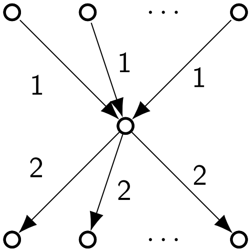

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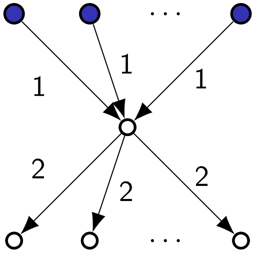

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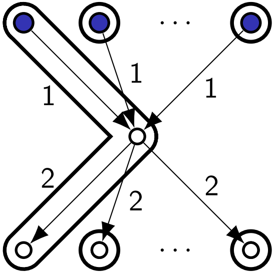

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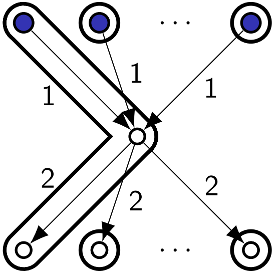

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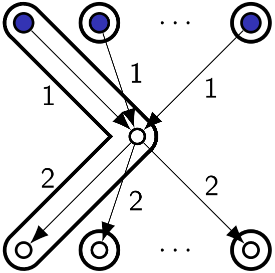

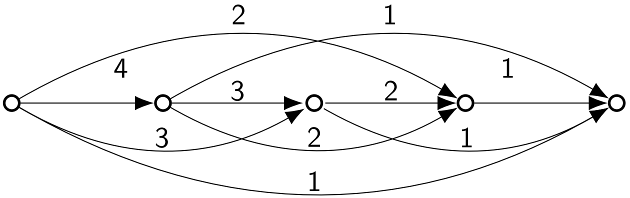

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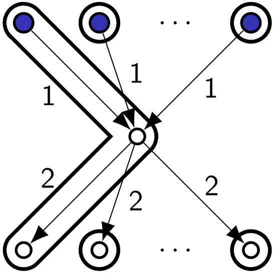

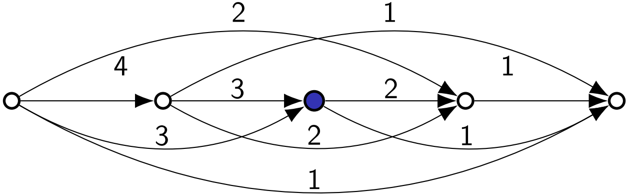

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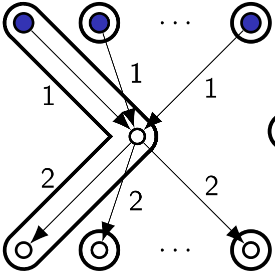

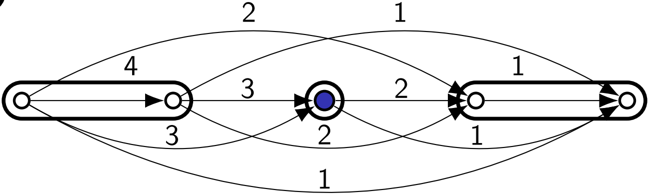

Parameterized complexity: reminder

Definition

For an input of size n with a parameter k and a computable function f:

- FPT algorithm $\Leftrightarrow f(k)n^{\mathcal{O}(1)}$
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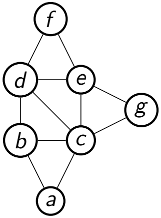
Commonly used method

Decomposing an input graph and applying dynamic programming

Parameterized complexity: tree decompositions

Tree decomposition of $G(V, E)$ [Halin, 1976] and others



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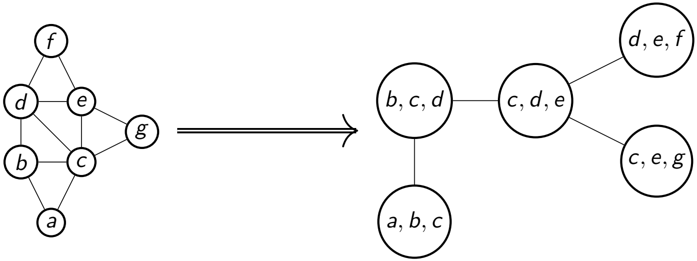

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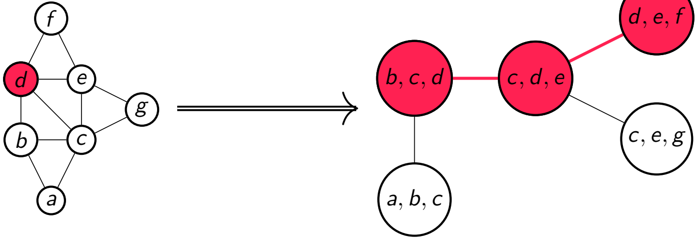

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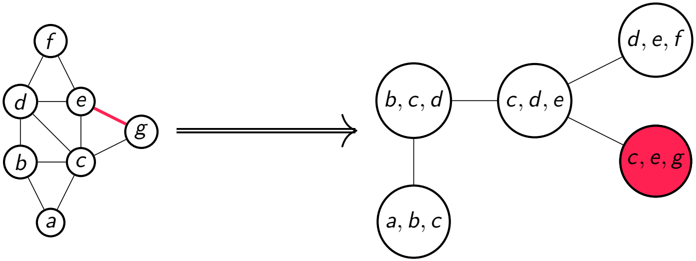

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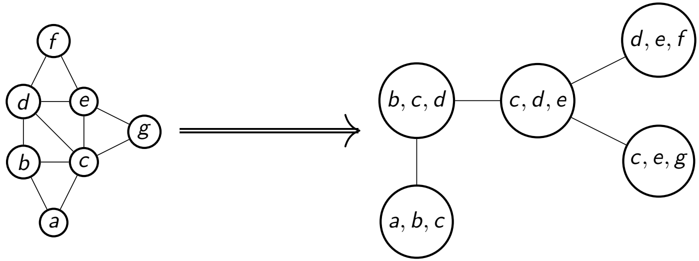

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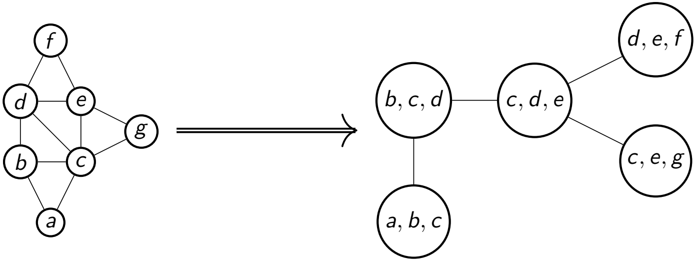

- Computation of a tree decomposition of width 2 tw in time $2^{\mathcal{O}(\text{tw})} n$ [Korhonen, 2021]

Parameterized complexity: nice tree decompositions

Nice tree decomposition of $G(V, E)$ [Kloks, 1994]

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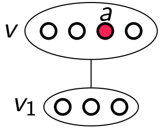

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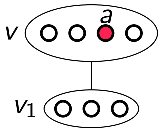

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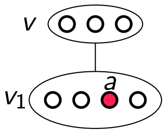

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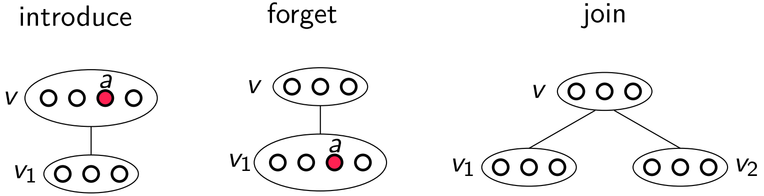

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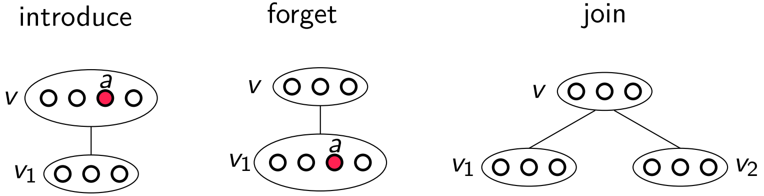

- From a tree decomposition, a nice one with same width and $\mathcal{O}(\text{tw}\, n)$ bags can be computed in time $\mathcal{O}(\text{tw}^2\, n)$ [Kloks, 1994]

Parameterized complexity for TD-PC (1) Preliminaries

Theorem [CDFK, 2024+]

TD-PC is FPT w.r.t. tw and t_{max} (total number of time-steps)

Dynamic programing on a nice tree decomposition

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Any arc of $\mathcal{D}$ appears in at most $t_{\max}$ paths of a TD-PC $\Rightarrow$ At most $p = \binom{\text{tw}}{2} \cdot t_{\max}$ temporally disjoint paths contain at least one arc from a given bag

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For simplicity, duplicate the arcs such that each has only one time label (so a TD-PC uses arc-disjoint paths)

Parameterized complexity for TD-PC (2) Type

Type: necessary information at each node v

$\Rightarrow$ At most
types for any node

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$\Rightarrow$ At most $p^p \times 2^{\mathrm{tw}+1} \times (\mathrm{tw}+1)! \times 2^{\mathrm{tw}+2} \times t_{\max}^2 \in 2^{\mathcal{O}(p \log p)}$ types for any node

Parameterized complexity for TD-PC (3) Consistency

Consistency of a type

- The ordered vertices V_i, the arcs of Q_i, and the information about the arcs going outside of X_v, induce temporal paths
- The arcs going outside of X_v exist in the digraph and their labels are compatible with the order
- Every vertex of X_v is in a V_i

Now, we compute from the bottom-up, maintaining consistency.

Parameterized complexity for TD-PC (4) Computation

Dynamic programing using consistent types of partial solutions

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And for TPC?

Same principle, but the paths can intersect $\Rightarrow$ More information in type: how many times in the solution does Q_i appear $\Rightarrow$ Running time $k^{\mathcal{O}(p \log p)} n$ where $k \in \mathcal{O}(n)$ is the solution size $\Rightarrow$ XP w.r.t. p

Conclusion and future work

Temporal class	**TPC**	**TD-PC**
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Rooted trees	$\mathcal{O}(\ell n^2)$	$\mathcal{O}(\ell n^2)$
Oriented trees	$\mathcal{O}(\ell n^2 + n^3)$	NP-hard
DAGs	NP-hard	NP-hard
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Perspectives

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