Path Covers of Temporal Graphs: When is Dilworth dynamic?

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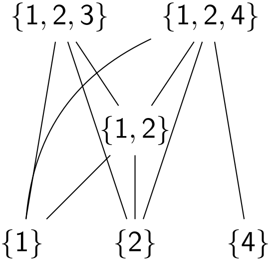
[3] LaBRI, Bordeaux

ANR GRALMECO and TEMPOGRAL

Introduction: Dilworth's theorem



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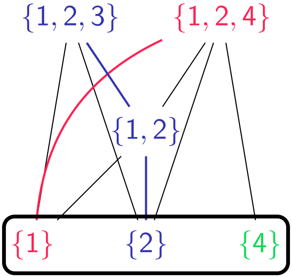
Theorem [Dilworth, 1950]

The minimum size of a chain partition of a finite poset is equal to the maximum size of an antichain of this poset.



Introduction: Dilworth's theorem

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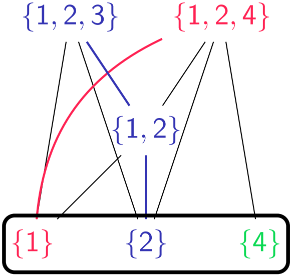
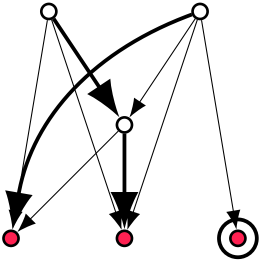
The minimum size of a path partition of a transitive DAG is equal to the maximum size of an antichain of this DAG.

Restated for graphs...





Introduction: Dilworth's theorem

Theorem [Dilworth, 1950]

The minimum size of a path cover of a DAG is equal to the maximum size of an antichain of this DAG.

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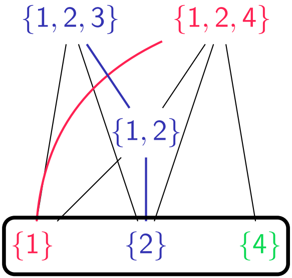

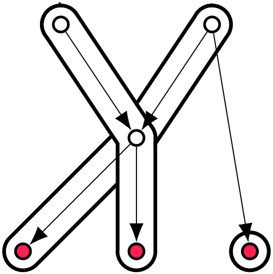

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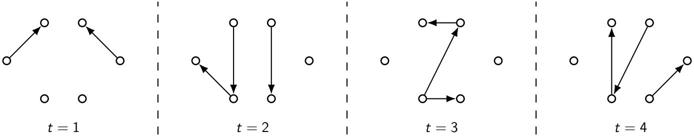
Algorithms:

- Algorithmic proof (polynomial time) [Fulkerson, 1956]
- Many improvements since then, now quasi-linear [Caceres, ICALP 2023]
- NP-hard on general graphs

Introduction: temporal (di)graphs

$\mathcal{D} = (V, A_1, A_2, \ldots, A_k)$ [Ferreira & Viennot, 2002]



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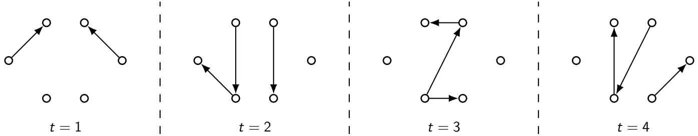
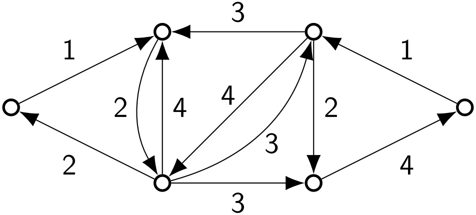


$\mathcal{D} = (V, A, \lambda)$ [Kempe, Kleinberg & Kumar, 2000]



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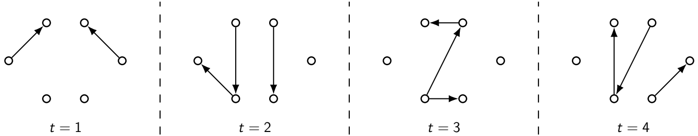

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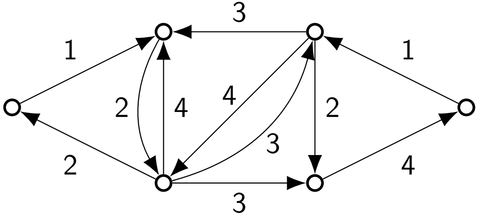
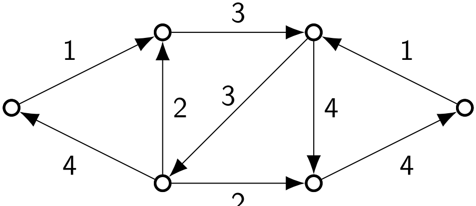


Many results and applications in distributed algorithms, dynamic networks (transportation, social, biological...). More recently, gain of interest from the graph algorithms community.

A few definitions for this talk

- A temporal DAG (resp. tree...) is a temporal (di)graph whose underlying (di)graph is a DAG (resp. tree...).



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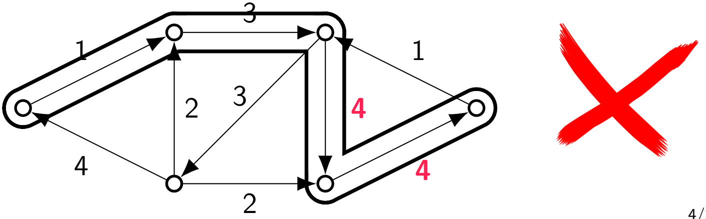

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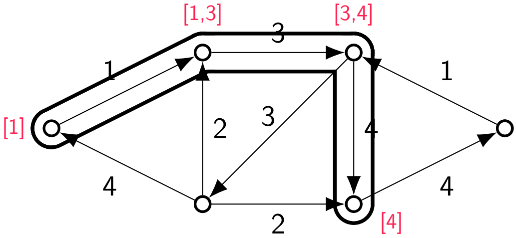

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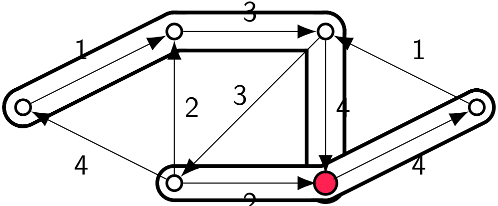

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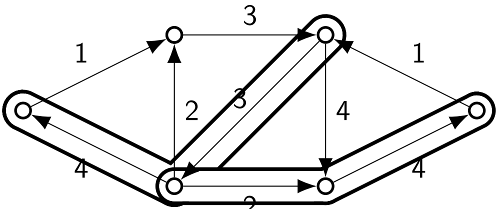

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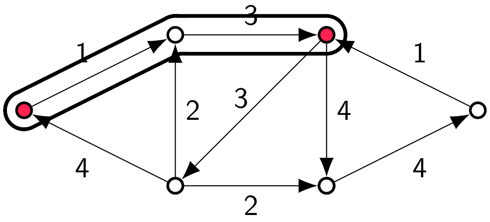

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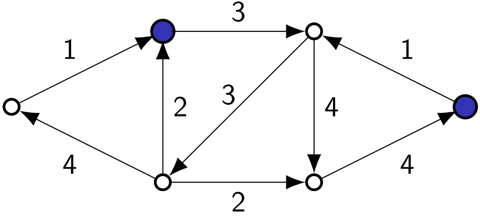
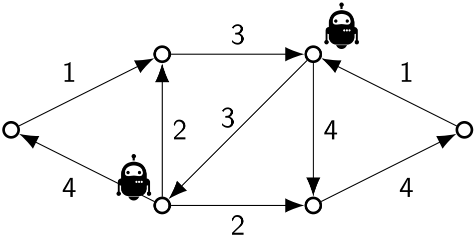
- Two vertices are temporally connected if there is a temporal path between them.
- A temporal antichain is a set of vertices who are pairwise not temporally connected.



Our incentive for temporally disjoint paths

History

- Several papers on paths and journeys in temporal graphs
- Temporally disjoint paths are a good model for dynamic MULTI AGENT PATH FINDING [Stern *et al.*, 2019]



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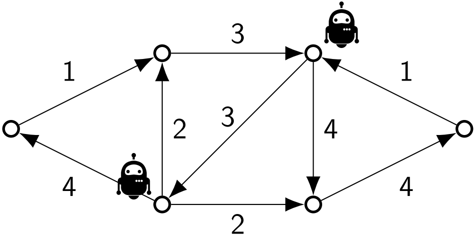

- TEMPORALLY DISJOINT WALKS: W[1]-hard and XP (number of walks) [Klobas *et al.*, IJCAI 2021]
- TEMPORALLY DISJOINT PATHS: NP-hard and W[1]-hard (number of vertices) on temporal stars [Kunz, Molter & Zehavi, IJCAI 2023]

A temporal Dilworth's theorem?

Dilworth property

In a (transitive) DAG, the minimum size of a path partition/cover is equal to the maximum size of a antichain.

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Two problems:

Temporal Path Cover (TPC)

Temporal Path Partition/Temporally Disjoint Path Cover (TD-PC)

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Temporal Path Cover (TPC)

Temporal Path Partition/Temporally Disjoint Path Cover (TD-PC)

Two questions:

Which temporal DAGs have the Dilworth property?

$\Rightarrow$ **Combinatorial** aspect

What is the complexity of those problems?

$\Rightarrow$ **Algorithmic** aspect

Our results

Temporal class	TPC	TD-PC
Oriented paths	$\mathcal{O}(\ell n)$	$\mathcal{O}(\ell n)$
Rooted trees	$\mathcal{O}(\ell n^2)$	$\mathcal{O}(\ell n^2)$
Oriented trees	$\mathcal{O}(\ell n^2 + n^3)$	NP-hard
DAGs*	NP-hard	NP-hard
Digraphs	XP (tw and $t_{\max}$) $n^{\mathcal{O}(\mathrm{tw}^2\, t_{\max} \log(\mathrm{tw}\, t_{\max}))}$	FPT (tw and $t_{\max}$) $2^{\mathcal{O}(\mathrm{tw}^2\, t_{\max} \log(\mathrm{tw}\, t_{\max}))} n$

* planar, subcubic, bipartite, girth 10, $\ell = 1$, $t_{\max} = 2$

$n =$ number of vertices
$\ell =$ number of (unsorted) time labels per arc
$t_{\max} =$ total number of time-steps

For those specific classes, polynomial-time $\Leftrightarrow$ Dilworth property.

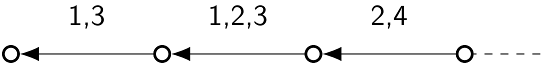
Temporal lines

Theorem [CDFK, 2024+]

Temporal oriented lines have the Dilworth property, and we can solve TPC and TD-PC in time $\mathcal{O}(\ell n)$.

Algorithm

Take a maximum-length temporal path containing a leaf.



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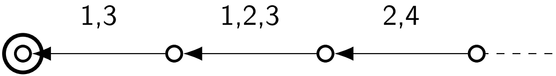

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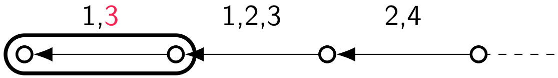

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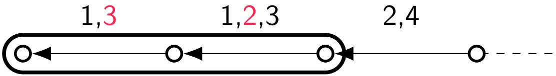

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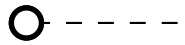

Iterate. The successive leaves are a temporal antichain!

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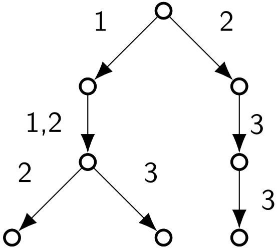

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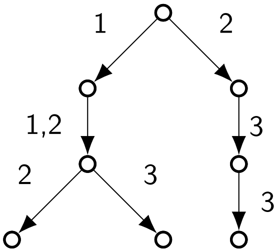
- Same principle as lines



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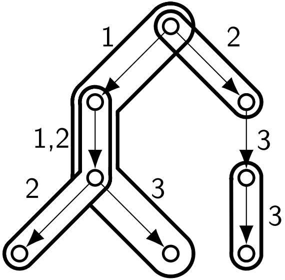

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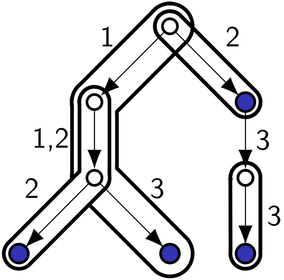

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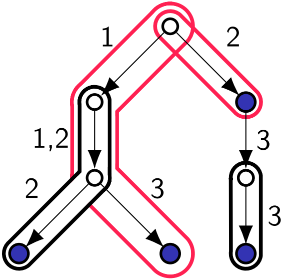

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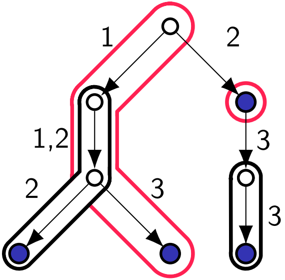

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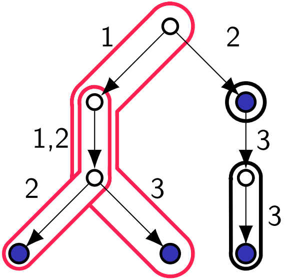

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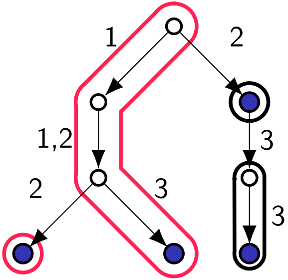
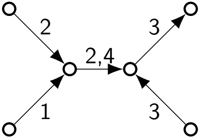


Path Cover of temporal oriented trees (1) Back to static

Theorem [CDFK, 2024+]

Temporal oriented trees have the Dilworth property for TPC, and we can solve TPC in time $\mathcal{O}(\ell n^2 + n^3)$.

$\mathcal{T} =$ 

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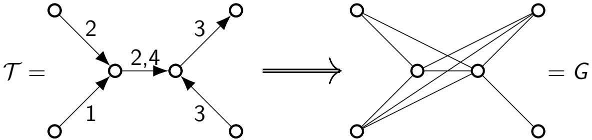

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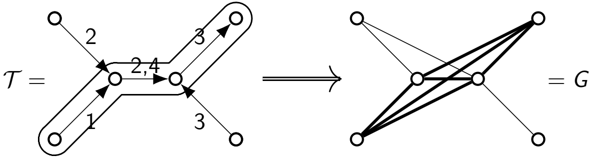

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Clique in G $\Leftrightarrow$ Temporal Path in $\mathcal{T}$.

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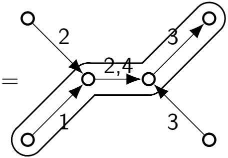
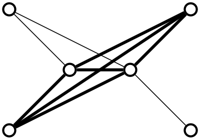
$\mathcal{T} =$  $\Longrightarrow$  $= G$

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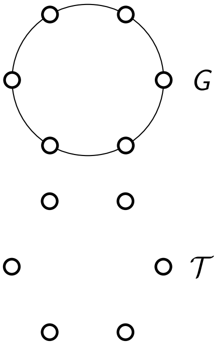

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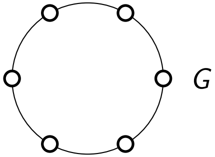

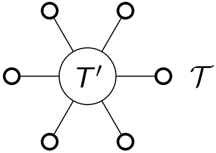

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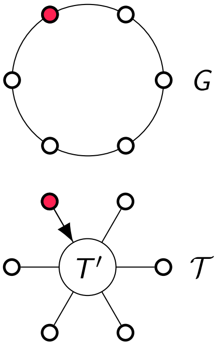

Path Cover of temporal oriented trees (2) Holes

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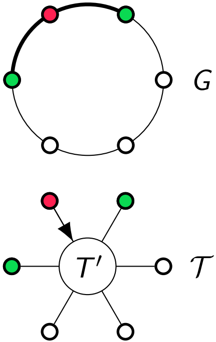

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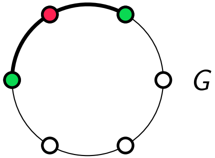

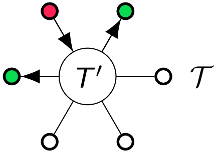

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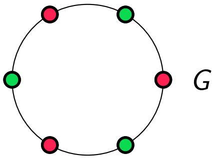

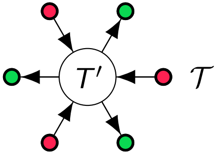

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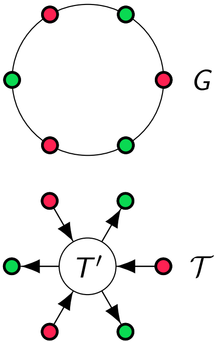

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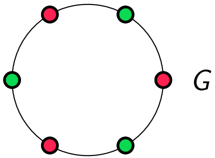
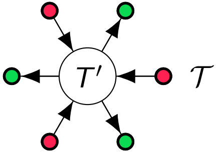
No even hole either (using Helly property and vertex-intersection of temporal paths).





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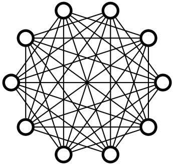
G 

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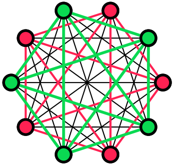

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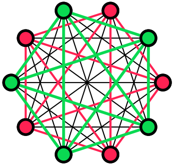
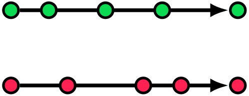
G  $\Longrightarrow$ $\mathcal{T}$ 

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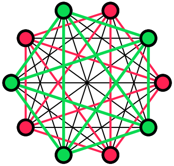
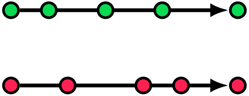
G  $\implies$ $\mathcal{T}$ 

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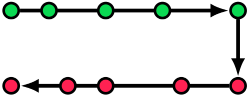

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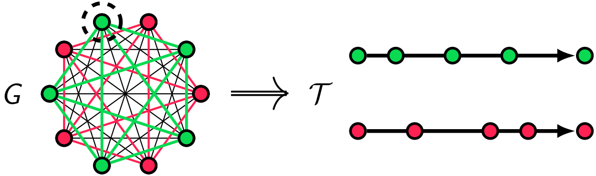

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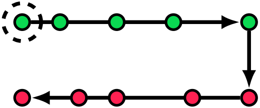

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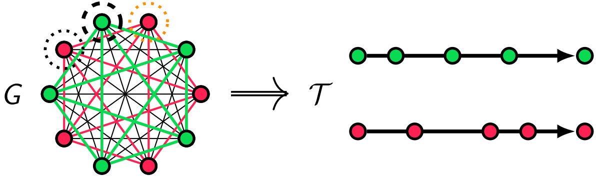

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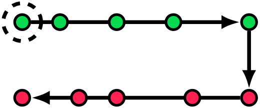

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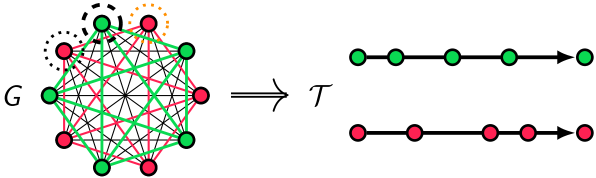

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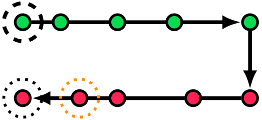

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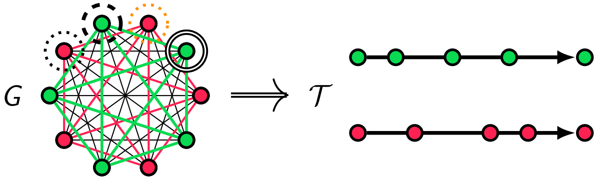

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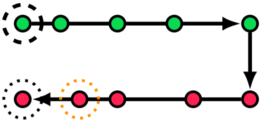

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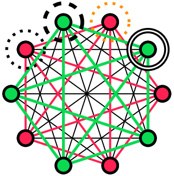
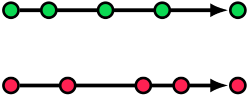
G  $\Longrightarrow$ $\mathcal{T}$ 

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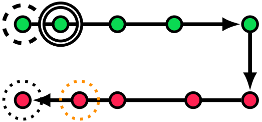

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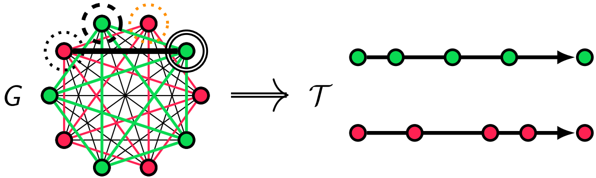

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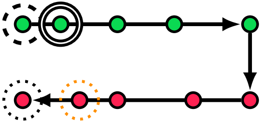

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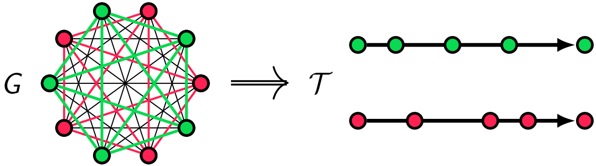

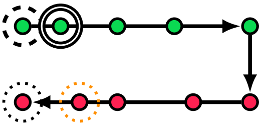

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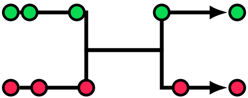

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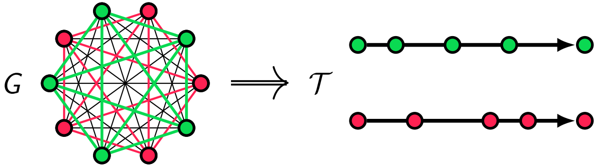

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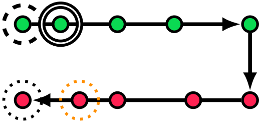

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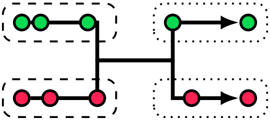

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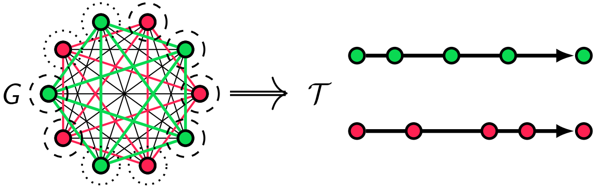

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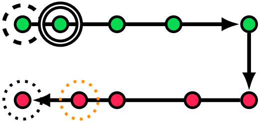

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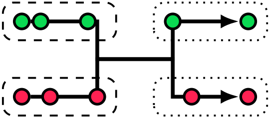

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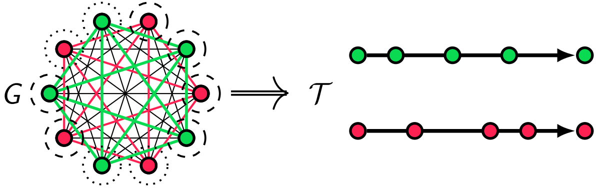

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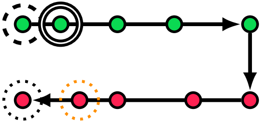

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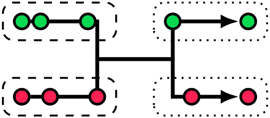

$\Rightarrow$ Each has to be complete bipartite in G, only possible if order $\leq$ 7, which we then manage.

Path Cover of temporal oriented trees (4) Conclusion

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The connectivity graph is (hole,antihole)-free

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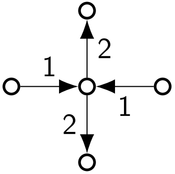

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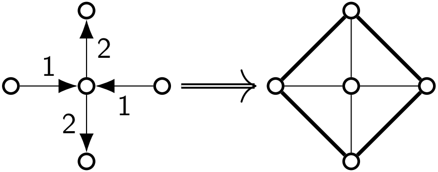

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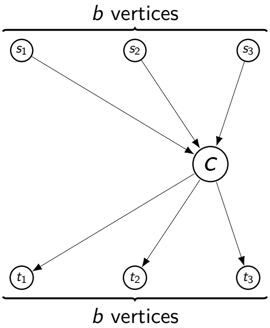

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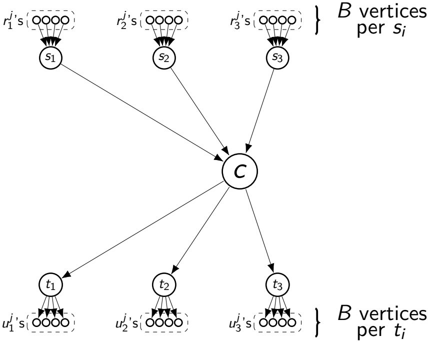

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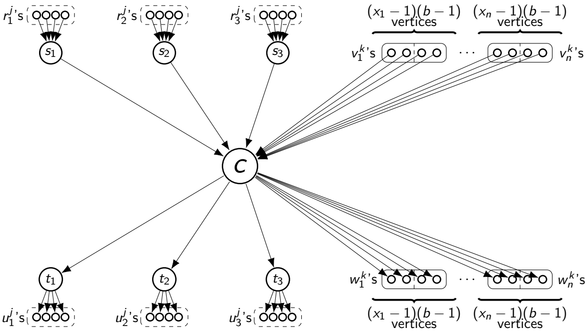

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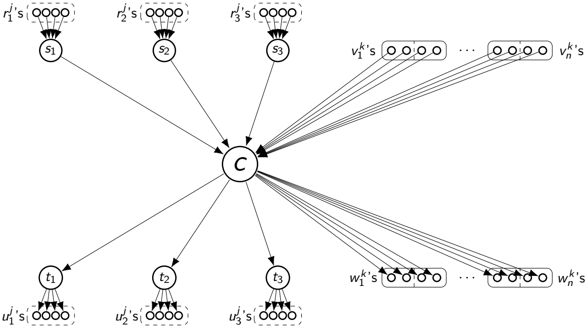

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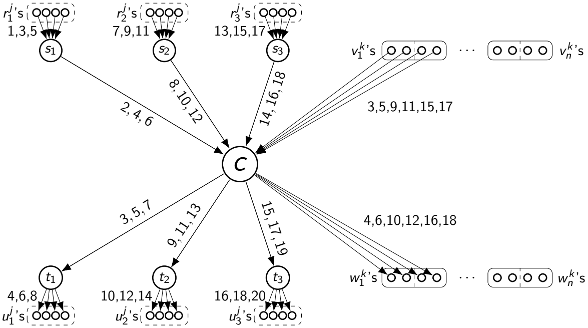

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Here, $x_1 = 3$

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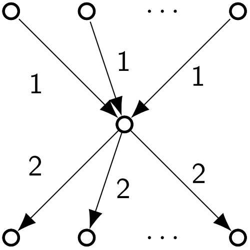

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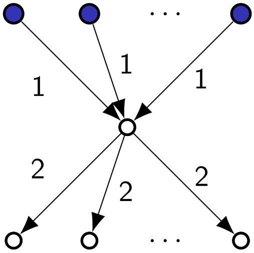

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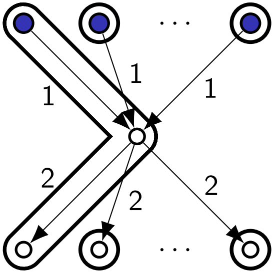

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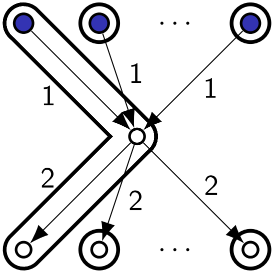

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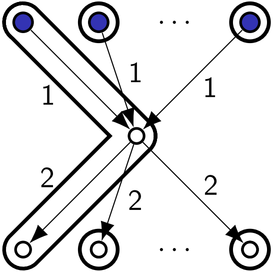

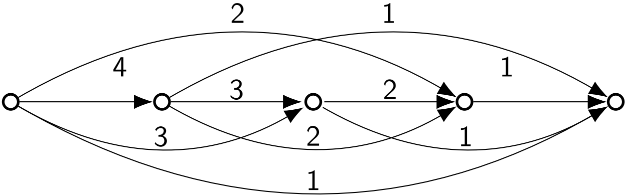

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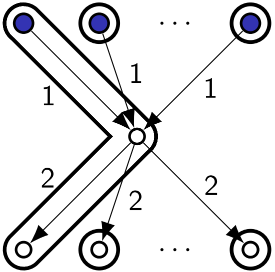

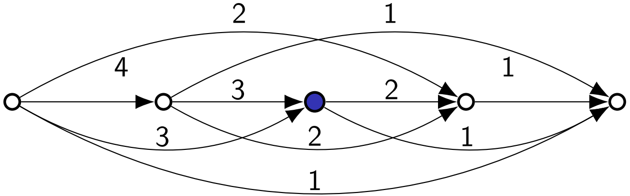

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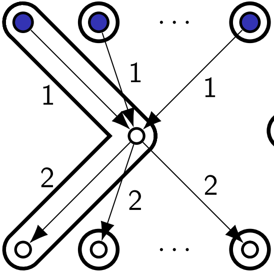

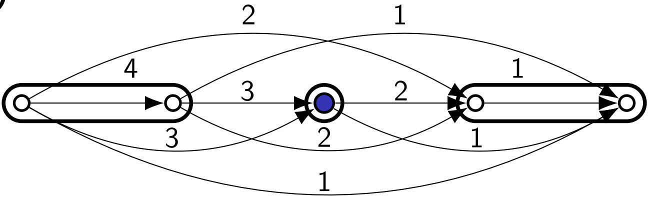

Parameterized complexity: reminder

Definition

For an input of size n with a parameter k and a computable function f:

- FPT algorithm $\Leftrightarrow$ $f(k)n^{\mathcal{O}(1)}$
- XP algorithm $\Leftrightarrow$ $n^{f(k)}$

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Nice tree decomposition of $G(V, E)$ [Kloks, 1994]

A rooted tree T, each node $v \in$ T has a *bag* $X_v \subseteq V$, such that:

- for $a \in V$, $\{v \ : \ a \in X_v\}$ is a connected subtree of T
- if $ab \in E$, then $\exists v$ such that $a, b \in X_v$
- *introduce* node $v \Leftrightarrow$ one child v_1 s.t. $X_v = X_{v_1} \cup a$, $a \in X_v$
- *forget* node $v \Leftrightarrow$ one child v_1 s.t. $X_v = X_{v_1} \setminus a$, $a \in X_{v_1}$
- *join* node $v \Leftrightarrow$ two children v_1, v_2 s.t. $X_v = X_{v_1} = X_{v_2}$

Leaf and root bags are empty, tw = max size of a bag -1

Parameterized complexity for TD-PC (1) Preliminaries

Theorem [CDFK, 2024+]

TD-PC is FPT w.r.t. tw and $t_{\max}$ (total number of time-steps)

Dynamic programing on a nice tree decomposition

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Observation

Any arc of $\mathcal{D}$ appears in at most $t_{\max}$ paths of a TD-PC $\Rightarrow$ At most $p = \binom{\mathrm{tw}}{2} \cdot t_{\max}$ temporally disjoint paths contain at least one arc from a given bag

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For simplicity, duplicate the arcs such that each has only one time label (so a TD-PC uses arc-disjoint paths)

Parameterized complexity for TD-PC (2) Type

Type: necessary information at each node v

$\Rightarrow$ At most
types for any node

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- For each Q_i, the vertices in V_i with one or two arcs outside of X_v, the time labels of those arcs, and whether the neighbour appears below or above v in the decomposition

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Parameterized complexity for TD-PC (3) Consistency

Consistency of a type

- The ordered vertices V_i, the arcs of Q_i, and the information about the arcs going outside of X_v, induce temporal paths
- The arcs going outside of X_v exist in the digraph and their labels are compatible with the order
- Every vertex of X_v is in a V_i

Now, we compute from the bottom-up, maintaining consistency.

Parameterized complexity for TD-PC (4) Computation

Dynamic programing using consistent types of partial solutions

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And for TPC?

Same principle, but the paths can intersect $\Rightarrow$ More information in type: how many times in the solution does Q_i appear $\Rightarrow$ Running time $k^{\mathcal{O}(p \log p)} n$ where $k \in \mathcal{O}(n)$ is the solution size $\Rightarrow$ XP w.r.t. p

Conclusion and future work

Temporal class	**TPC**	**TD-PC**
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Rooted trees	$\mathcal{O}(\ell n^2)$	$\mathcal{O}(\ell n^2)$
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DAGs	NP-hard	NP-hard
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