The Closed Geodetic Game: algorithms and strategies

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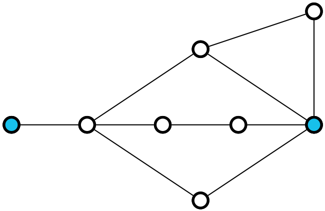
Geodetic Sets

Geodetic closure [Harary & Nieminen, 1981]

For a set S of vertices: the set of all vertices in shortest paths between vertices of S, denoted by $\langle S \rangle$.



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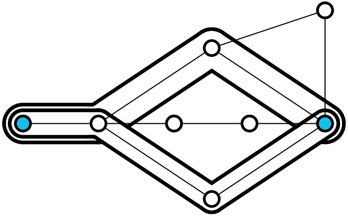

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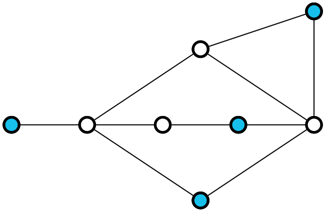
A set S of vertices of graph $G(V, E)$ such that $(S) = V$.



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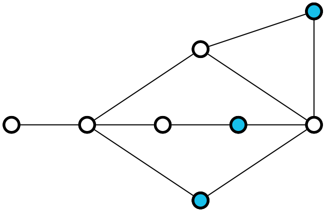

- Many combinatorial and algorithmic results...

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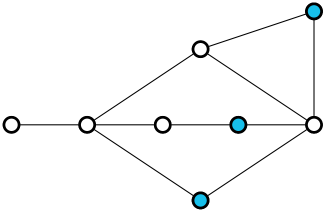

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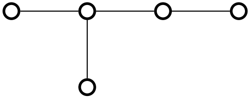

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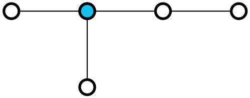

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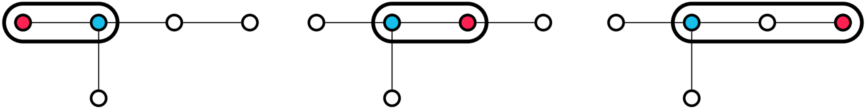

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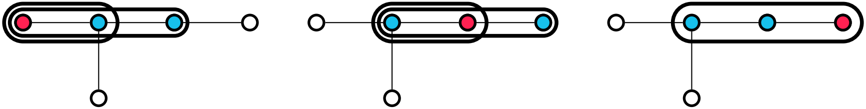

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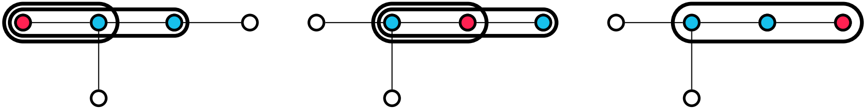

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- Generalized wheels [Nečásková, 1993]
- Complete multipartite graphs, hypercubes, graphs with a unique optimal geodetic set [Haynes, Henning & Tiller, 2003]

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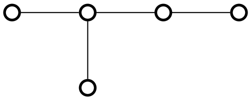

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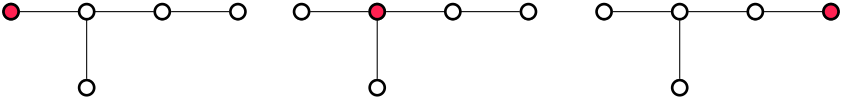

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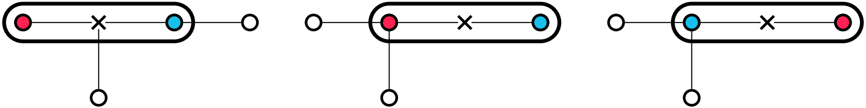

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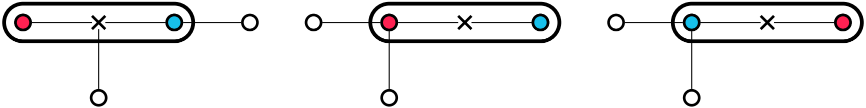

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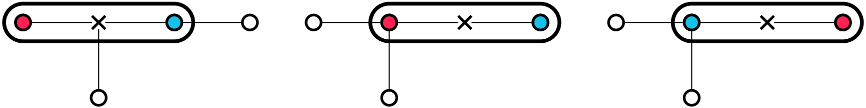

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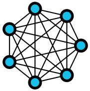
→ We study the CLOSED GEODETIC GAME

First results: Grundy values

Some trivial ones

- $\mathcal{G}(K_n) = n \bmod 2$ (every vertex has to be selected)



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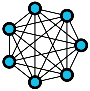

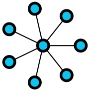

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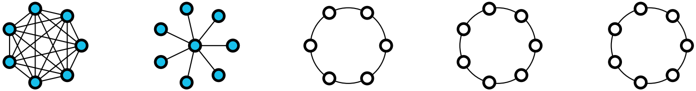

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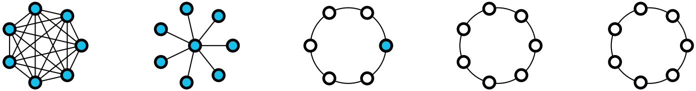

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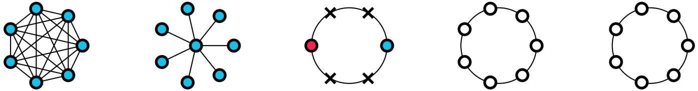

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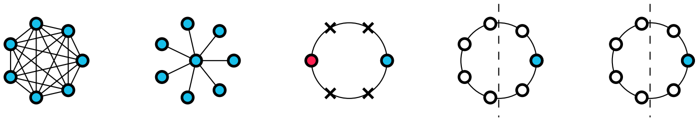

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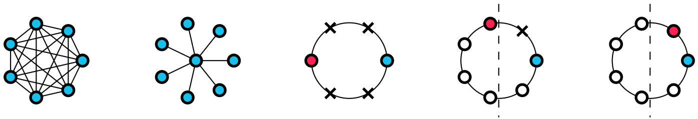

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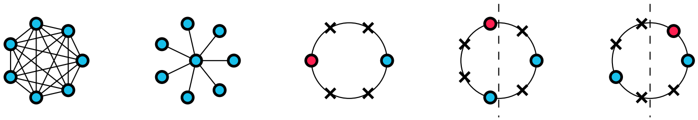

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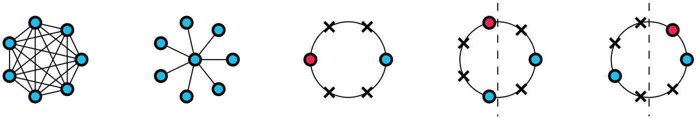


Some less-trivial ones

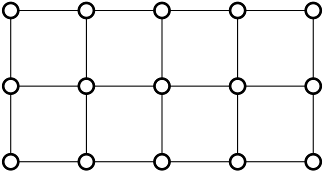
- $\mathcal{G}(P_n) = n \bmod 2$ (the value is expected, but the proof is nontrivial!)
- $\mathcal{G}(K_{m,n}) = 0$ if m and n have the same parity, and 2 otherwise

A fun result: grids

Proposition

A multidimensional grid has outcome $\mathcal{N}$ if and only if all its dimensions are odd.

Strategy



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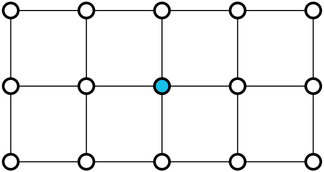

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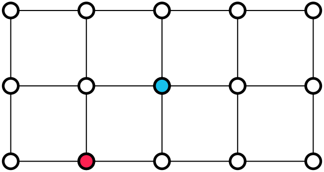
- Afterwards:



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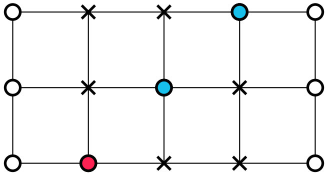
- Afterwards: symmetry strategy!



Product results

Theorem [D., Gahlawat & Myint, 2024+]

For the Cartesian, tensor and strong products, the outcome of the product is $\mathcal{N}$ if and only if the outcomes of the two graphs are $\mathcal{N}$.

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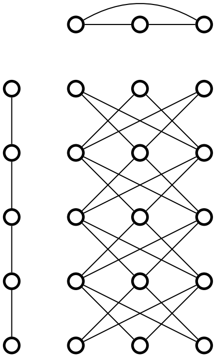
Proof idea (for tensor product)

Assume G and H are $\mathcal{N}$.



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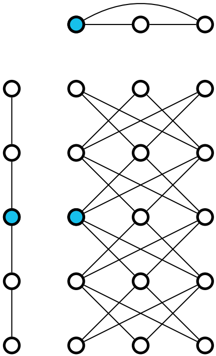

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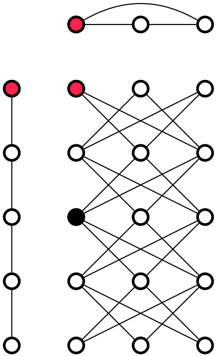
- Same row/column



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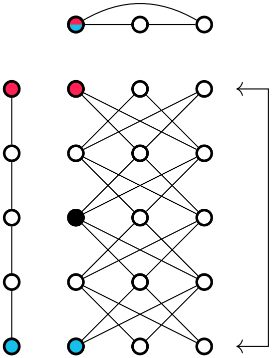

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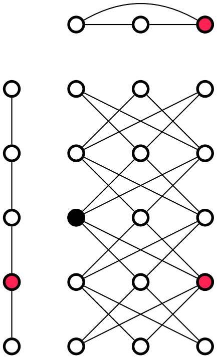

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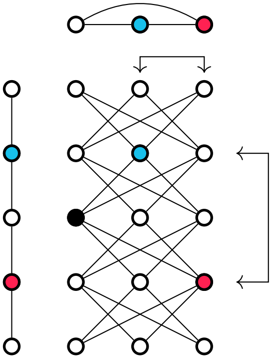

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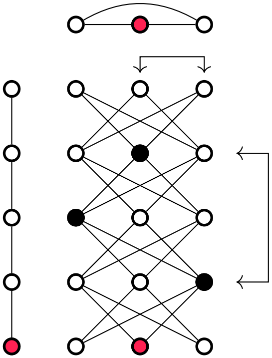

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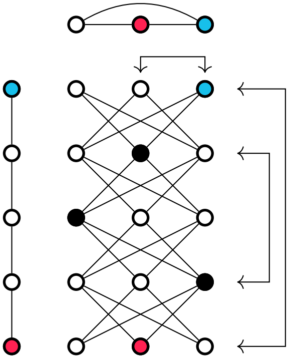
- Move on associated rows/columns $\Rightarrow$ Answer on associated, associate new rows/columns



Algorithms for Grundy values

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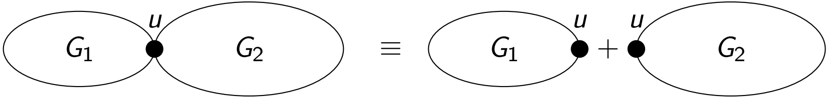
Lemma

If u is an articulation point linking maximal components $G_1, \ldots, G_k$, then:

$$G, \{u\} \equiv (G_1, \{u\}) + \ldots + (G_k, \{u\}).$$



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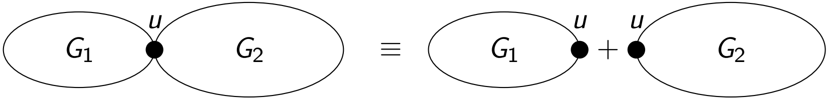

In a tree, every vertex is either a leaf or an articulation point $\Rightarrow$ Apply dynamic programing to compute the Grundy value

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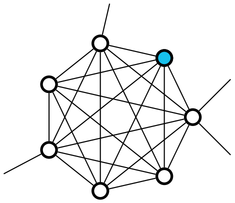

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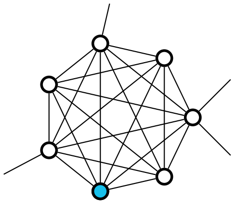

Algorithms for Grundy values: block graphs

Theorem [D., Gahlawat & Myint, 2024+]

There is a linear-time algorithm computing the Grundy values of block graphs.

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- All non-articulation points moves on a given clique are equivalent
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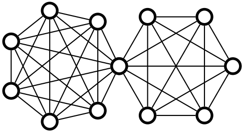

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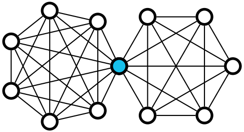

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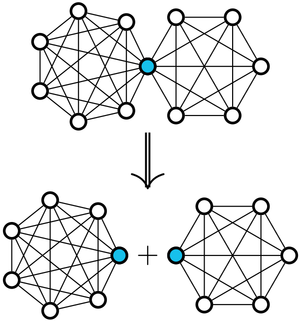

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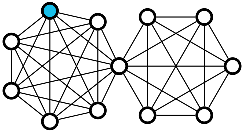

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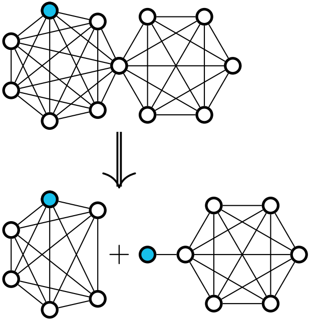

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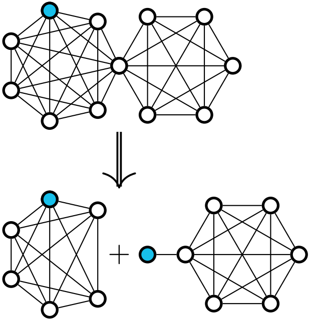

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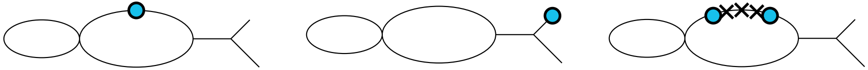

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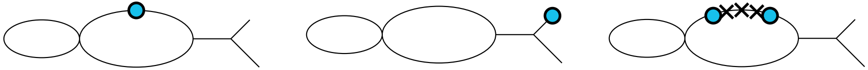

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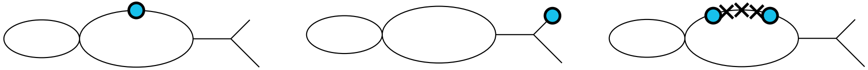

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- DP algorithms for Grundy values extending the ideas for trees

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