Strengthening the Murty-Simon Conjecture on diameter-2-critical graphs

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Recall the fundamentals: distance, diameter

Distance

The distance between two vertices is the number of edges in a shortest path between them.

Diameter

The diameter of a graph is the highest distance between two of its vertices.

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Theorem (Moon, Moser, 1966)

Almost all random graphs have diameter 2.

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Proof idea

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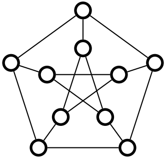
$\Rightarrow \mathbb{E}(diam(G) > 2) = \binom{n}{2}\frac{1}{2}\left(\frac{3}{4}\right)^{n-2} \xrightarrow[n \to +\infty]{} 0$

Diameter-2-critical (D2C) graphs

Definition

A graph is D2C if it has diameter 2



Diameter-2-critical (D2C) graphs

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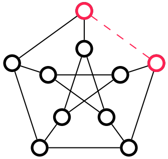
A graph is D2C if it has diameter 2 and if any edge deletion increases its diameter.



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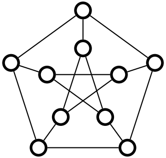
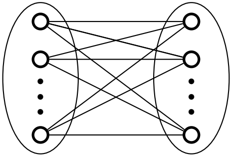
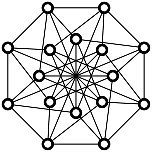
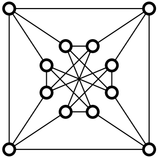




Complete bipartite graphs



Clebsch Graph



Chvàtal Graph

Broader context: DdC graphs

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$\Rightarrow$ The main focus is on D2C graphs

How many edges are there in a D2C graph?

- Diameter 2 $\Rightarrow$ Dense
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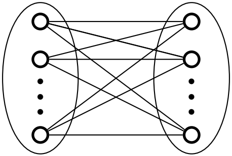

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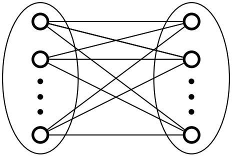

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Example: D2C graphs with no triangle

At most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges; equality $\Leftrightarrow G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ (Mantel, 1907).

The Murty-Simon Conjecture

Conjecture (Murty, Simon, Ore, Plesník, 1970s)

If G is a D2C graph of order n, then, it has at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, with equality iff $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

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Early history

- True for D2C graphs with no triangle (Mantel, 1907)
- $m < \frac{3n(n-1)}{8} = 0.375(n^2 - n)$ (Plesník, 1975)

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- $m < 0.2532n^2$; true for $n \leq 24$, $n = 26$ (Fan, 1987)

Related graph classes

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Maximal triangle-free (MTF) graphs

- A graph is MTF if it is triangle-free and adding an edge creates a triangle.
- A triangle-free graph is D2C iff it is MTF. (Barefoot *et al.*, 1995)

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3-total-domination-critical (3TC) graphs

- A graph is 3TC if it has total domination number 3 and adding an edge reduces it.
- A graph is D2C iff its complement is 3TC (or 4-supercritical) (Hanson and Wang, 2003)

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- A graph is 3TC if it has total domination number 3 and adding an edge reduces it.
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- Several graph classes for which the Conjecture holds, among which:
  - G has a dominating edge (Hanson and Wang, 2003; Haynes *et al.*, 2011; Wang, 2012)
  - $\Delta \geq 0.6756n$ (Jabalameli *et al.*, 2016+)
  - $\Delta < 0.6756n$ and less than $(\frac{5}{14} + o(1))n$ edges in a triangle (D. and Hansberg, 2018)

Some related results

- Plesník, 1986:
  - There exist infinitely many D2C graphs with every edge in a triangle and minimum degree d (for $d \geq 2$)
  - There exist infinitely many planar D2C graphs with every edge in a triangle and minimum degree 2
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- Loh and Ma, 2016:
  - There is an infinite family of D2C graphs with average edge-degree at least $(\frac{10}{9} - o(1))n$
  - There are c, N such that every D2C graph of order at least N has average edge-degree at most $(\frac{6}{5} - c)n$
  - If $d \geq 3$, then the average edge-degree of a DdC graph is at most n, and the bound is tight
  - Every DdC graph has at most $\frac{3n^2}{d}$ edges; every DdC ($d \geq 3$) graph has at most $\frac{n^2}{6} + o(n^2)$ edges

A breakthrough

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There exists an n_0 such that, for every $n > n_0$, the Murty-Simon Conjecture holds for D2C graphs of order n.

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- $n_0 \approx 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \Big\}$ size 10^{14}
- First use of the Regularity Lemma to get an exact, non-asymptotic value
- Proof idea: a D2C graph with more than $(\frac{1}{4} - o(1))n^2$ is almost complete bipartite

Strengthening the Conjecture?

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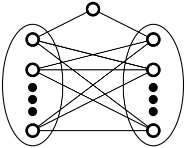
Claim (Füredi, 1992)

If G is D2C and non-bipartite, then, it has at most $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1 \approx \left\lfloor \frac{n^2}{4} - \frac{n}{2} \right\rfloor$ edges, with equality iff G is obtained by subdividing and edge of $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.



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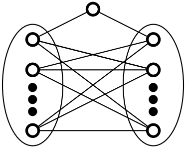

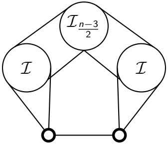

Theorem (Balbuena *et al.*, 2015)

If G is D2C, non-bipartite and triangle-free, then, it has at most $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ edges, with equality iff G is an inflation of C_5.

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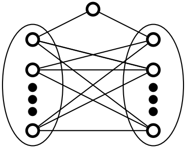

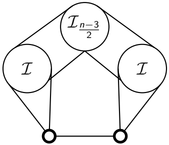

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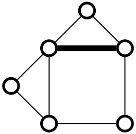
Conjecture: linear strengthening (Balbuena *et al.*, 2015)

If G is D2C and non-bipartite, and $G \neq H_5$, then, it has at most $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ edges.

$H_5 =$ 

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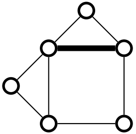
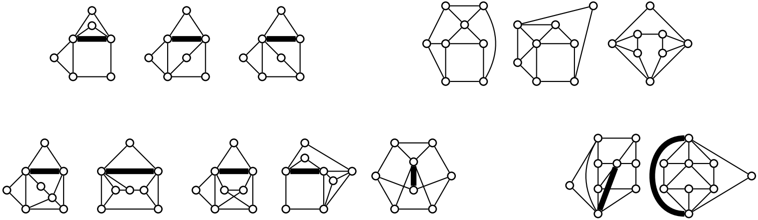
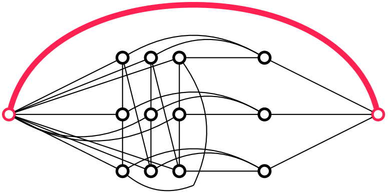
$H_5 =$ 

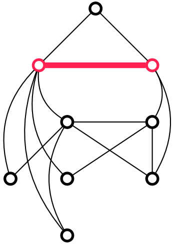


Our main result

Theorem (D., Foucaud, Hansberg, 2019)

If G is D2C non-bipartite with a dominating edge and $G \neq H_5$, then it has at most $\left\lfloor \frac{n^2}{4} \right\rfloor - 2$ edges.





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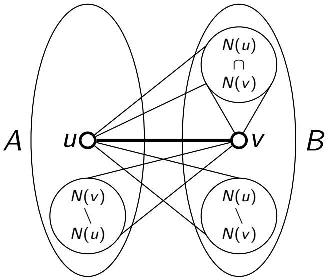

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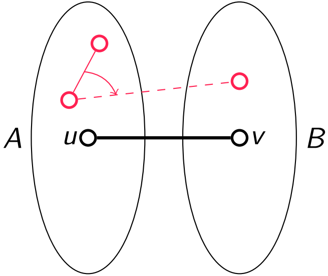

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Sketch of the proof

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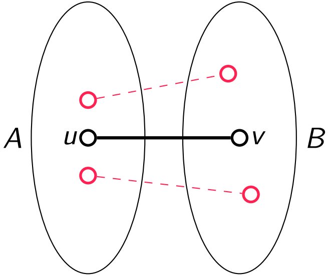
2. Assign every edge in A and B to a non-edge between them.
3. Find two non-assigned non-edges between A and B.



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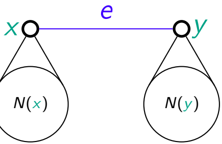

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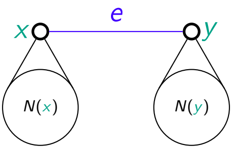

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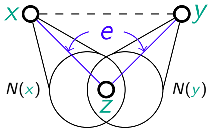

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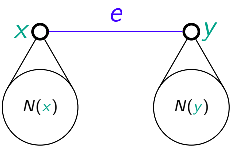

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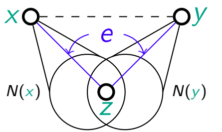

$\Rightarrow$ Every edge is critical for a pair of vertices

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The function f

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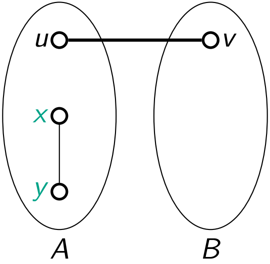

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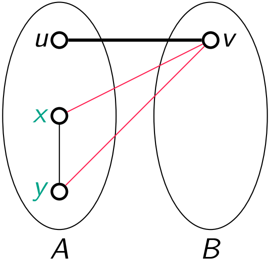

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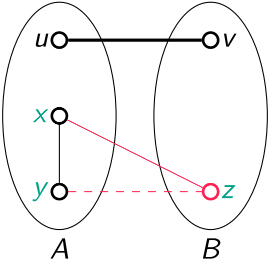
- Critical for y and z with $z \in B \cap N(x)$



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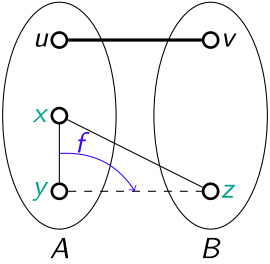

Assigning internal edges to external non-edges

The function f

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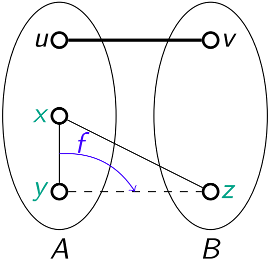

Lemma

f is injective.

Non-assigned non-edges?

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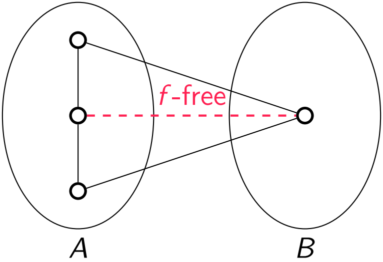

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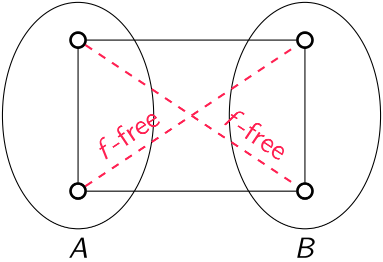

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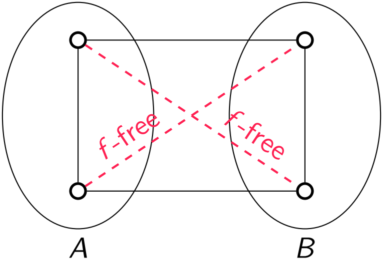

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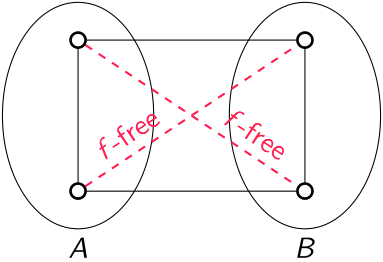

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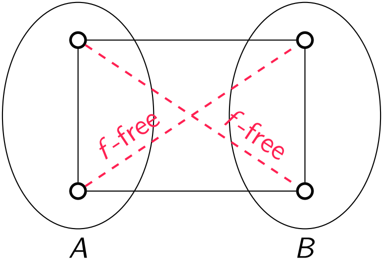

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$\Rightarrow$ This proves the bound of Murty-Simon

The next step

Sketch of the proof

1. Partition the vertices in two sets A and B.
2. Assign every edge in A and B to a non-edge between them.
3. Find two non-assigned non-edges between A and B.

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$\rightarrow$ For the remainder of the proof, we may assume that
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Defining an orientation of the internal edges

We denote $S_u = N(u) \setminus \{v\}$ and $S_v = N(v) \setminus \{u\}$.

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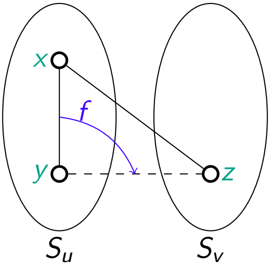

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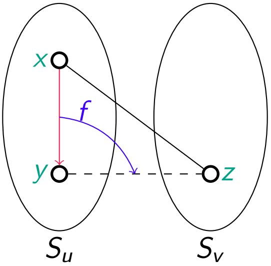

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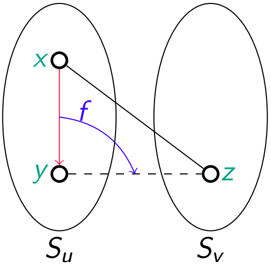


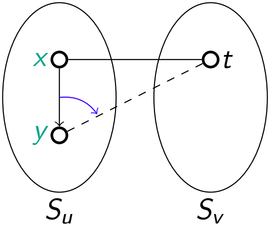
The next (and final!) step

Find properties of the f-orientation and prove that there are 2 f-free non-edges.

A useful property

Lemma

Let $\overrightarrow{xy}$ be an arc of the f-orientation in (wlog) S_u such that neither x nor y is incident to an f-free non-edge.



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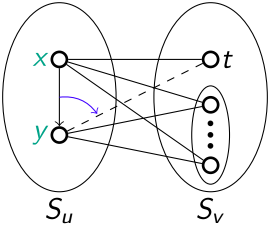

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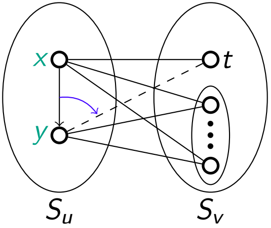


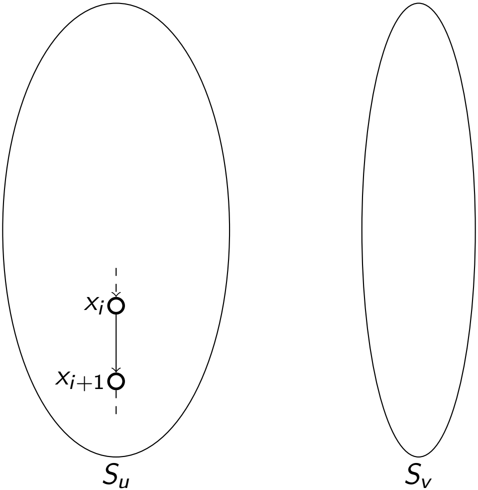
Corollary

Let $\overrightarrow{C}$ be a directed cycle of the f-orientation. Then, there is at least one f-free non-edge incident with the vertices of $\overrightarrow{C}$.

Directed cycles are even stronger!

Lemma

Let $\overrightarrow{C}$ be a directed cycle of the f-orientation. Then, there are at least $|\overrightarrow{C}|$ f-free non-edges incident with the vertices of $\overrightarrow{C}$.



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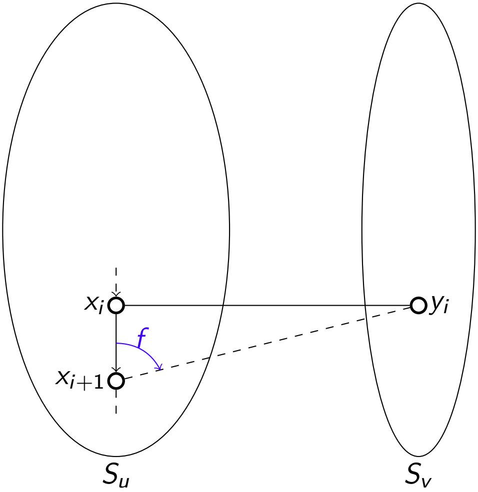

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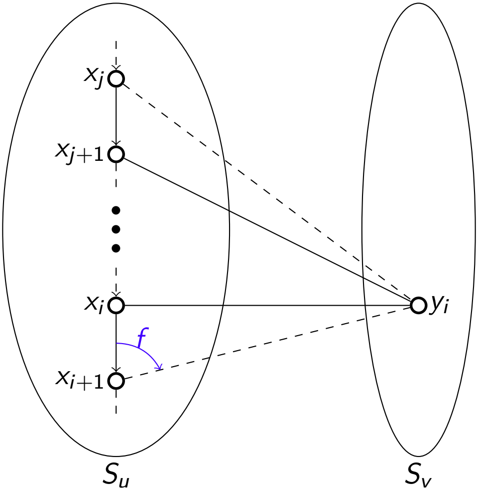

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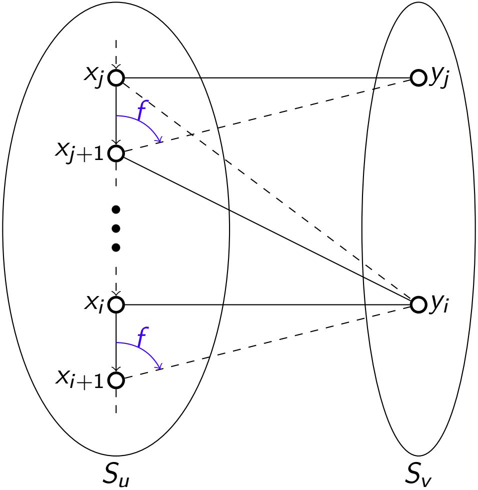

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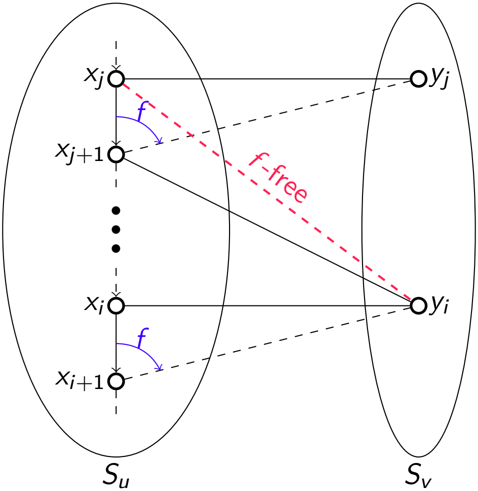
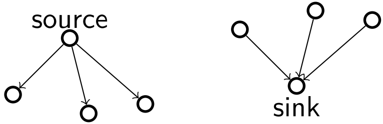


End of the proof

No directed cycle
$\Rightarrow$ At least one
source and one sink



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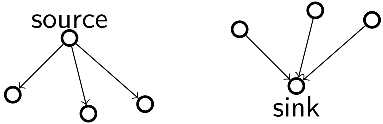

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Every source and every sink has at least one f-free non-edge in its closed neighbourhood.

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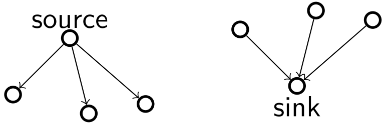

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⇒ Proves Murty-Simon for this family!

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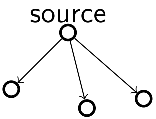

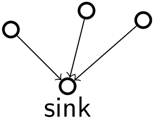

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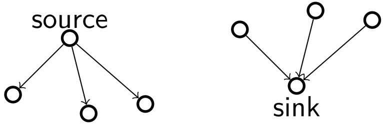

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- Otherwise ⇒ Refining even more the properties of the f-orientation to conclude

Stronger results (under conditions)

Theorem (D., Foucaud, Hansberg, 2019)

If uv is only critical for the pair $\{u, v\}$, then G has at most $\left\lfloor \frac{n^2}{4} \right\rfloor - c_1 - 2c_2$ edges (c_1 (resp. c_2) = number of components of diameter 2 (resp. ≥ 3) in the graph induced by S_x ($x \in \{u, v\}$) oriented by the f-orientation).

Theorem (D., Foucaud, Hansberg, 2019)

If uv is only critical for the pair $\{u, v\}$, then G has at most $\left\lfloor \frac{n^2}{4} \right\rfloor - \sum_{C \in \mathcal{C}} |C| - |\mathcal{S}|$ edges ($\mathcal{C}$ (resp. $\mathcal{S}$) = directed cycles (resp. transitive triangles and disjoint neighbourhoods of a source or a sink) in the graph induced by S_x ($x \in \{u, v\}$) oriented by the f-orientation).

Au final

Conclusion

- Improving on the bound of Murty-Simon
- Better proof for this family
- The f-orientation, which might be useful

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