Rooks and ARC-KAYLES

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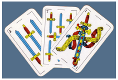
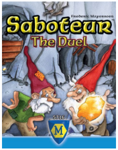
Thanks to Nicolas Bousquet for his help.

Seminario Preguntón, December 13, 2017

Combinatorial Games

Definition

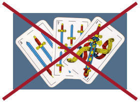


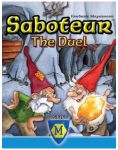


Combinatorial Games

Definition

1. Two-player games





Combinatorial Games

Definition

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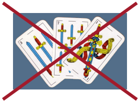
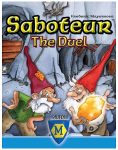
2. No chance

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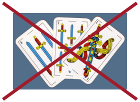


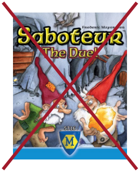

Combinatorial Games

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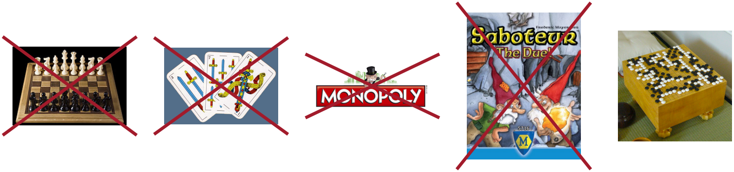

Combinatorial Games

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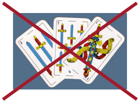
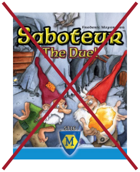
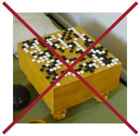
5. The last move alone determines the winner



Combinatorial Games

Relaxations

1. Two-player games $\rightarrow$ Multiplayer Theory
2. No chance $\rightarrow$ Economical Games
3. Perfect information $\rightarrow$ Economical Games
4. Finite games, no draw $\rightarrow$ Loopy Games
5. The last move alone determines the winner $\rightarrow$ Scoring Games, several graph games (coloring game, domination game, ...)

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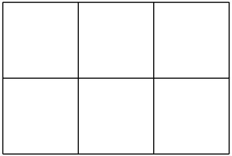
$\rightarrow$ We will talk about pure Combinatorial Games.
Note: Both players play perfectly!

Let's play!

CRAM: The players place dominos on a grid. The player who plays the last domino wins.



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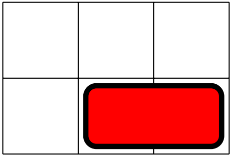

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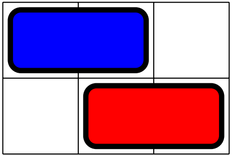

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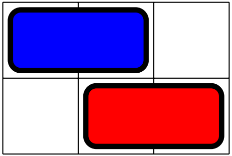

$\Rightarrow$ Second player wins.

Let's play!

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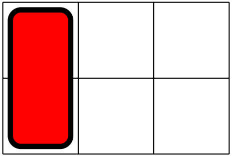

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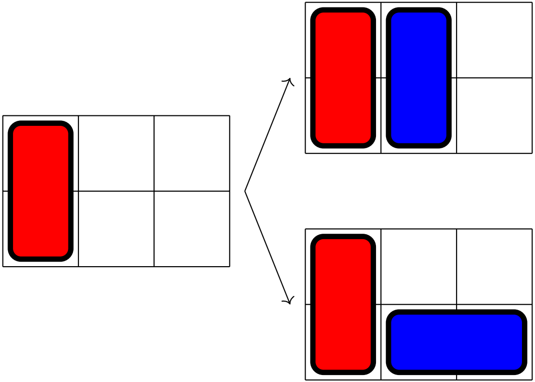

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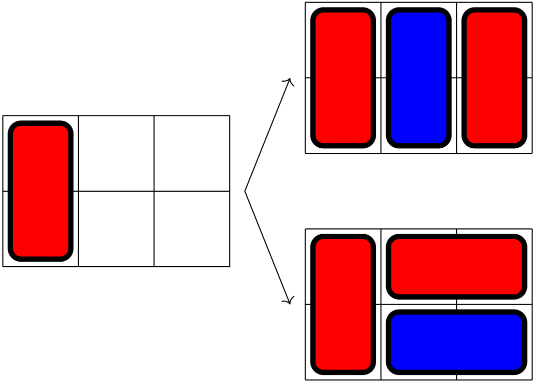

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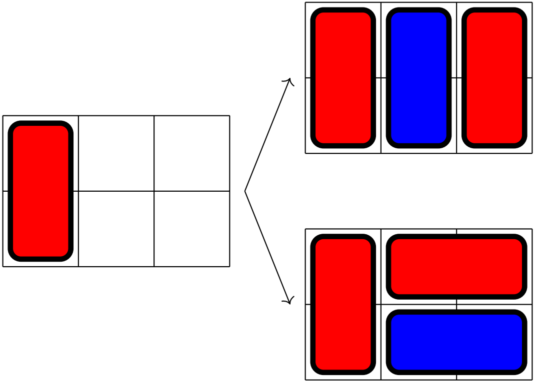

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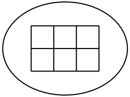
Problematics of Combinatorial Games

1. Is a given game $\mathcal{N}$ or $\mathcal{P}$?
2. What is the winning strategy?

An inefficient algorithm: the game graph



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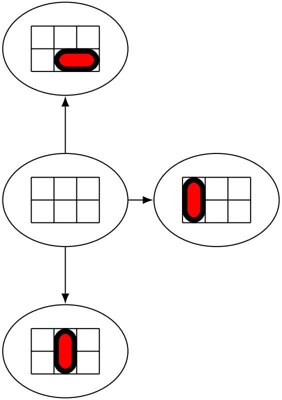

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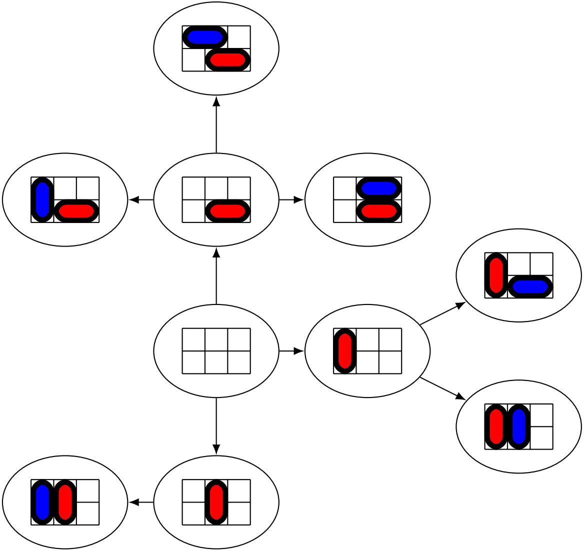

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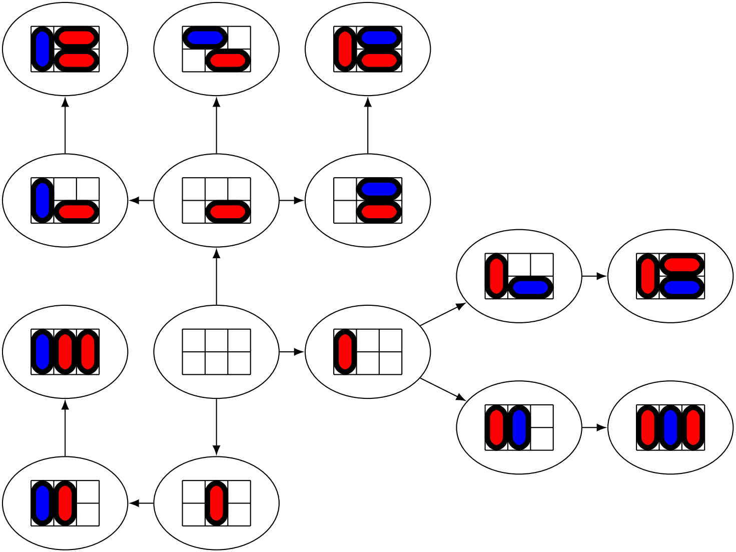

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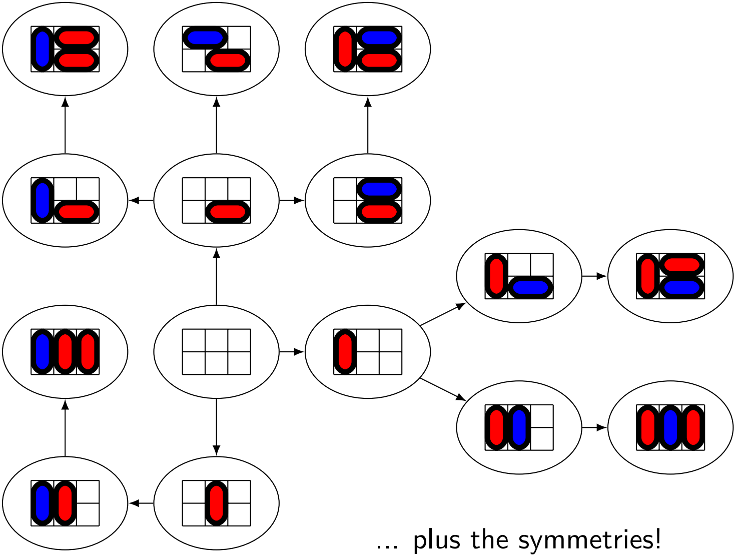

… plus the symmetries!

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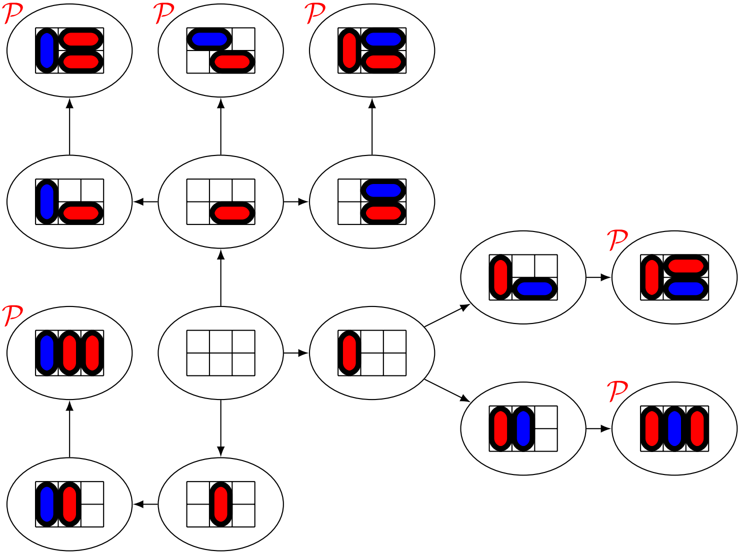

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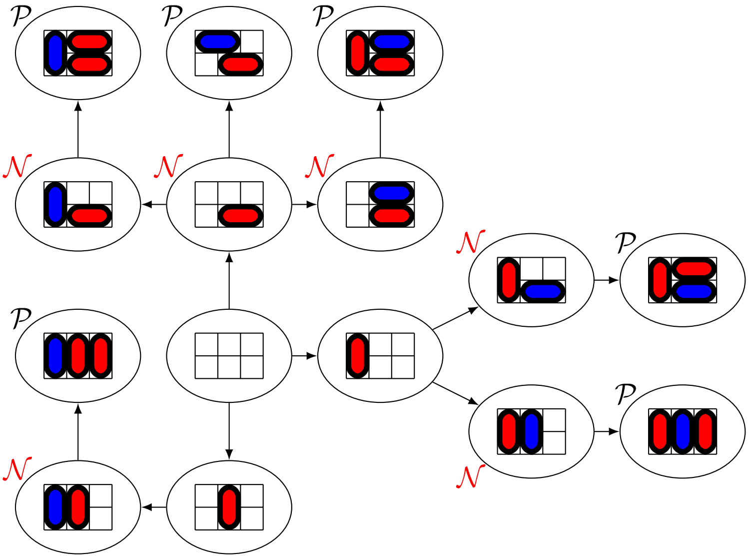

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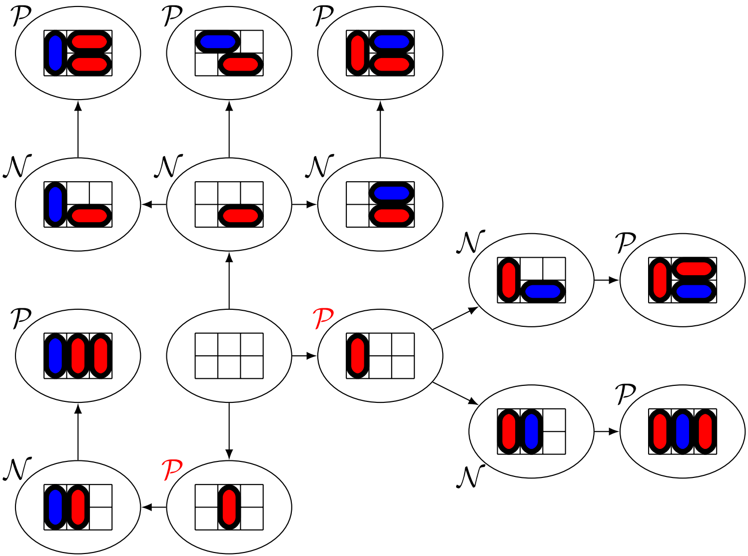

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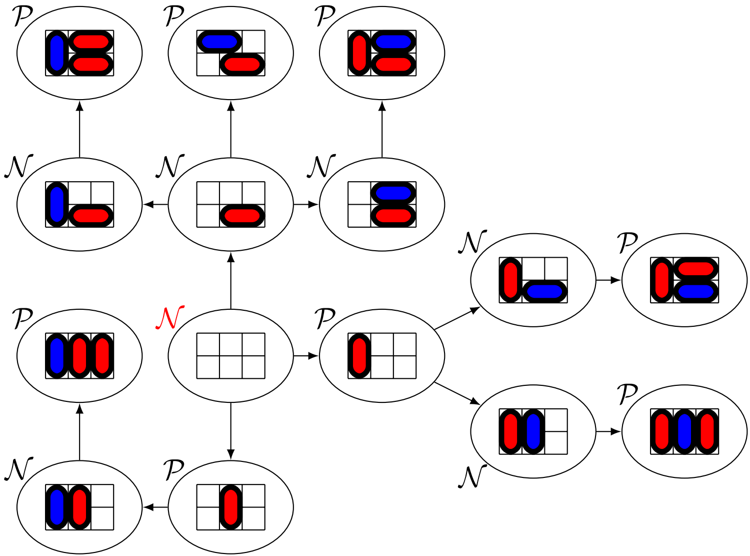

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- ... but exponential-time!

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$\Rightarrow$ A more efficient method to study games

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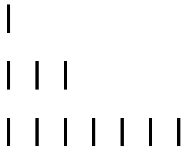

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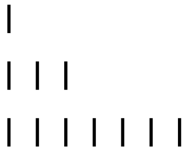

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$\Rightarrow$ Is there a strategy?

Solving Nim

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It is $\mathcal{P} \Leftrightarrow a_1 \oplus \ldots \oplus a_n = 0$.

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|	1
| | |	3
| | | | | | |	7

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Back to CRAM: summing games

Back to Cram: summing games

Sum of games

On $G + H$, the players play either on G or on H. When G (resp. H) is over, they play on H (resp. G).
The player who plays the last move wins.

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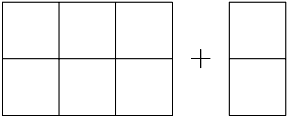

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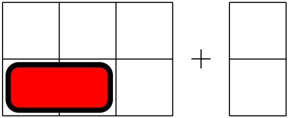

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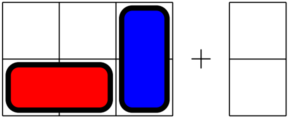

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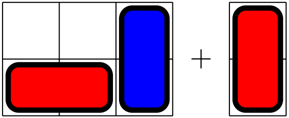

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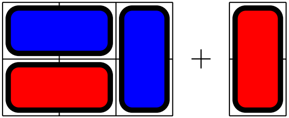

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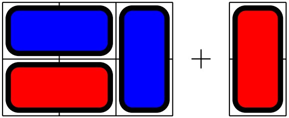

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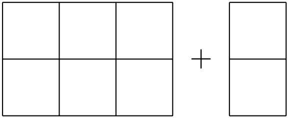

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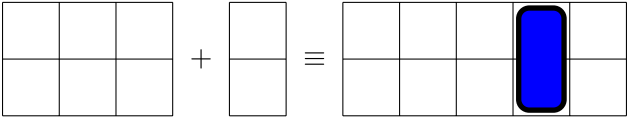

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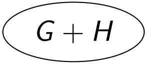

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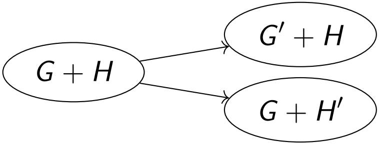

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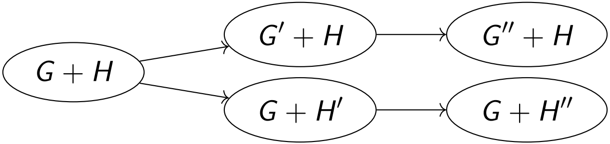

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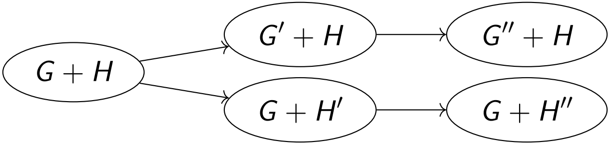

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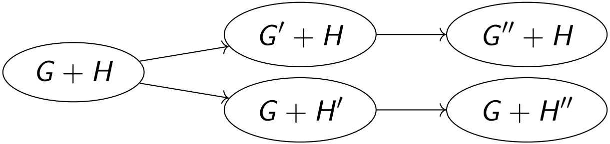

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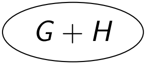

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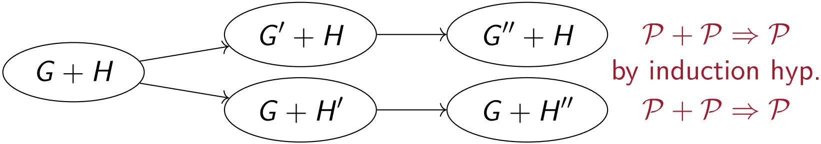

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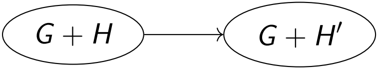

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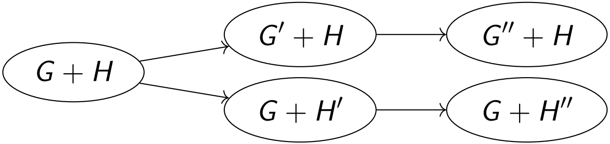

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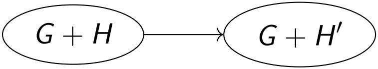

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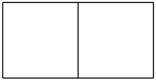

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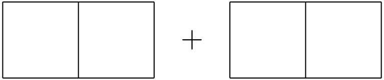

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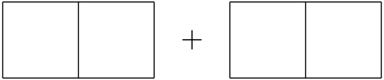

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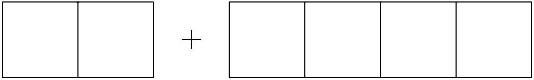

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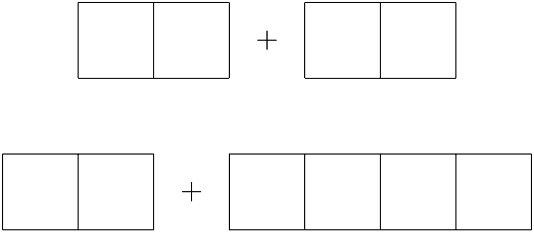

$\Rightarrow \mathcal{P}$

$\Rightarrow \mathcal{N}$

$\Rightarrow$ We need to define equivalence classes for games

Grundy values

Equivalence of games

$G \equiv H \Leftrightarrow G + H$ is $\mathcal{P}$.

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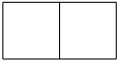

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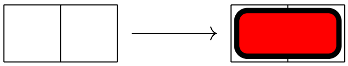

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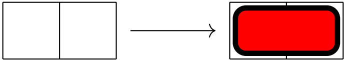

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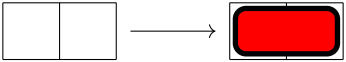
$\mathcal{G} = 1$  $\mathcal{G} = 0$

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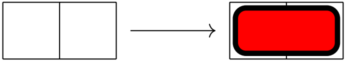
$\mathcal{G} = 1$  $\mathcal{G} = 0$

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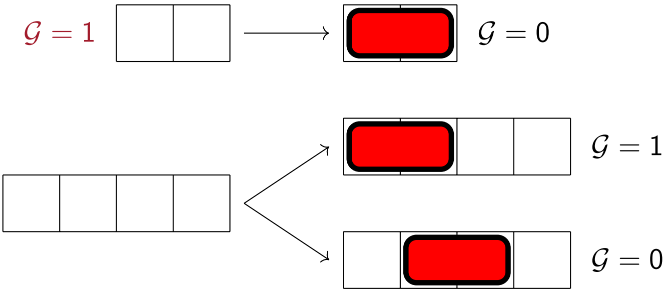

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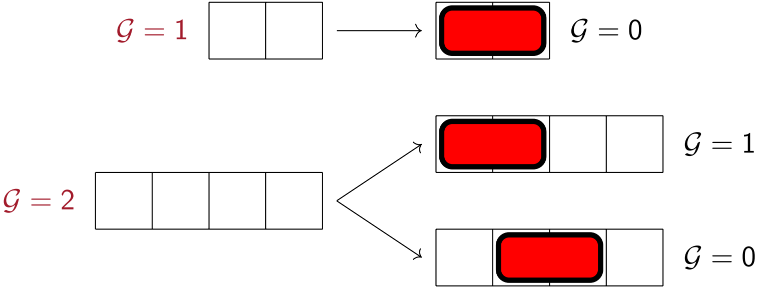

Sprague-Grundy Theorem

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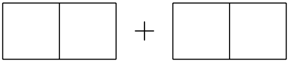

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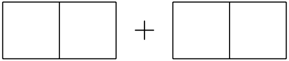

$$1 \oplus 1$$

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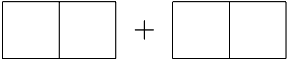

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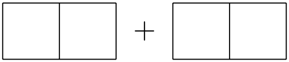

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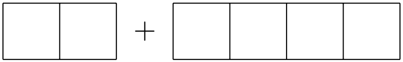

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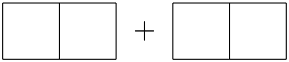

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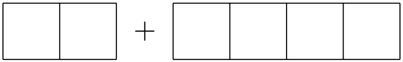

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Interpretation

Every impartial game is equivalent to a NIM heap.

The Rooks game: motivation

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The queens problem (Bezzel, 1848)

How many queens can one place on a chessboard without them attacking each other?

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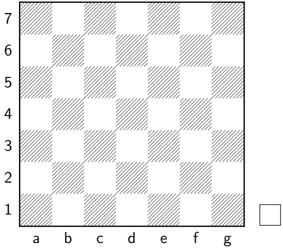
→ Solved on odd square chessboards.



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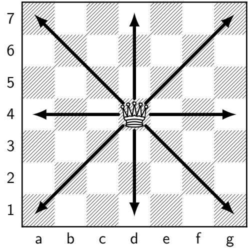

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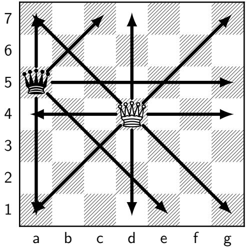

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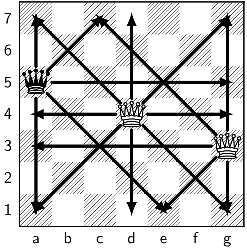

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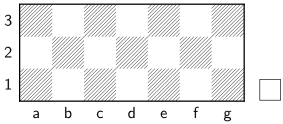

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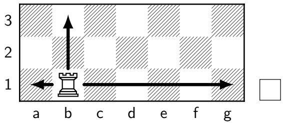

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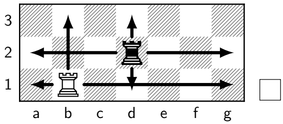

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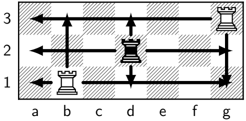

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Same than the rooks game, but the chessboard has holes. Rooks cannot attack through holes.

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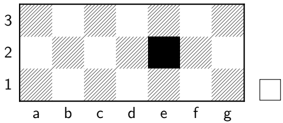

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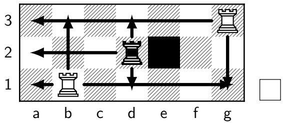

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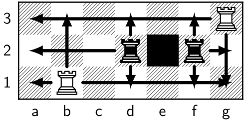
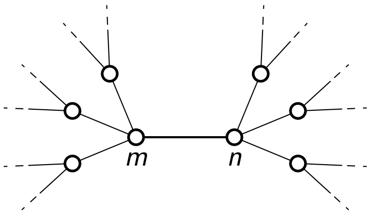


A model for the rooks game: Weighted Arc-Kayles

Weighted Arc-Kayles (or WAK)

- Played on a weighted graph $G = (V, E, \omega)$ with $\omega : V \to \mathbb{N}$.
- The players alternate selecting edges.
- The weight of both endpoints is decreased by 1.
- Vertices with weight zero are removed.
- When there are no edges left, the game ends.



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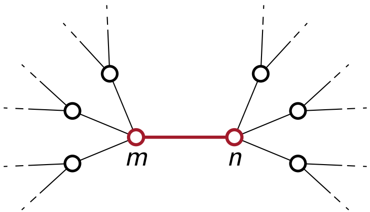

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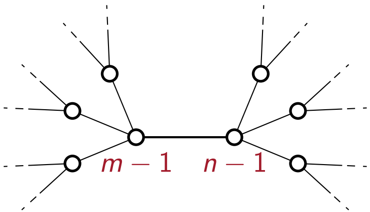
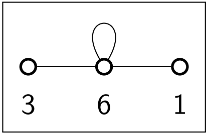


Example



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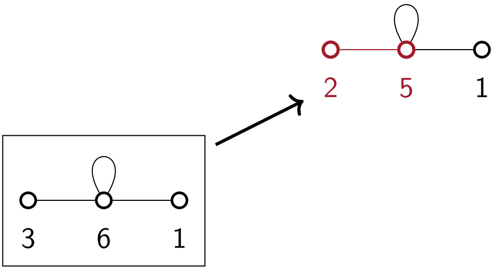

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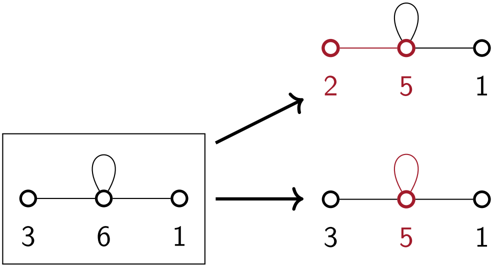

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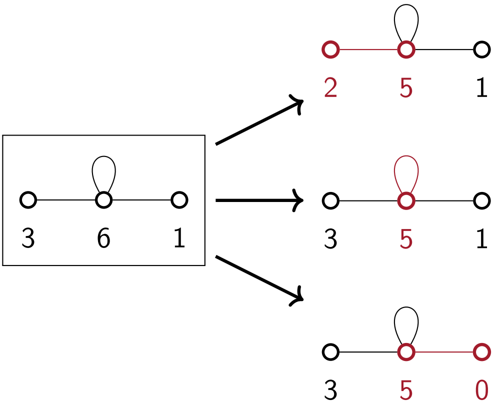

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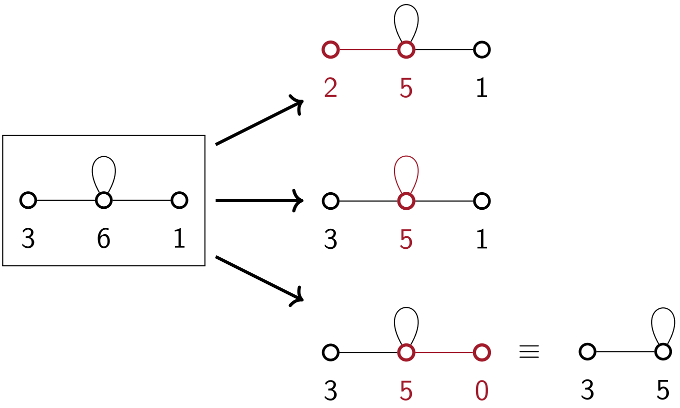

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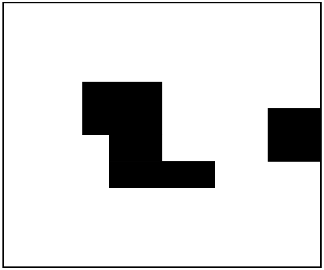

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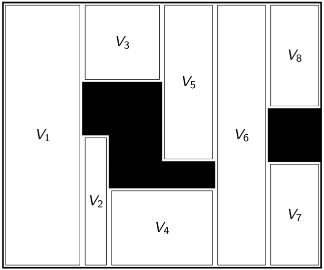

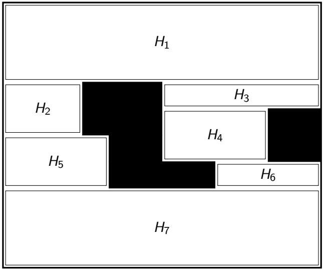

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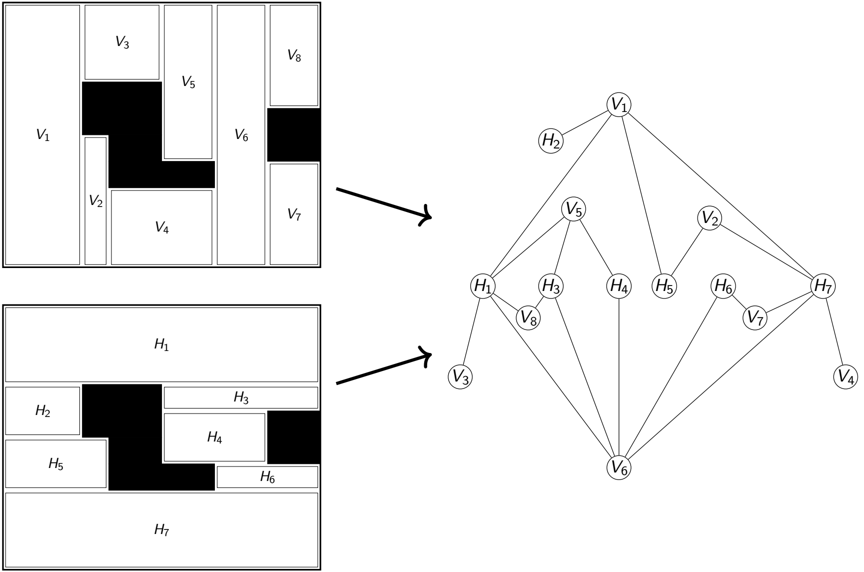

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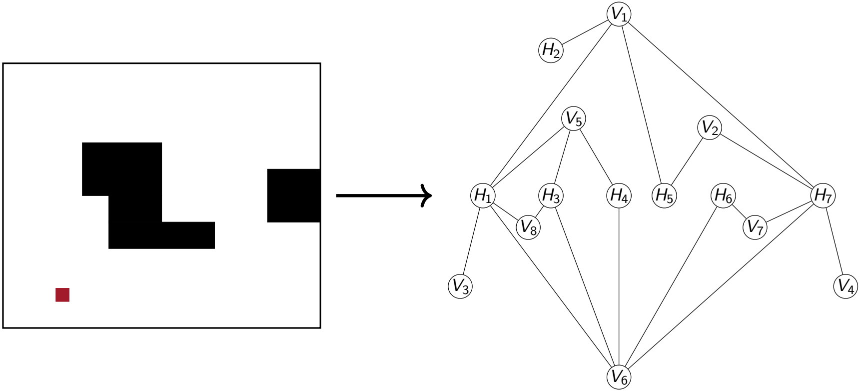

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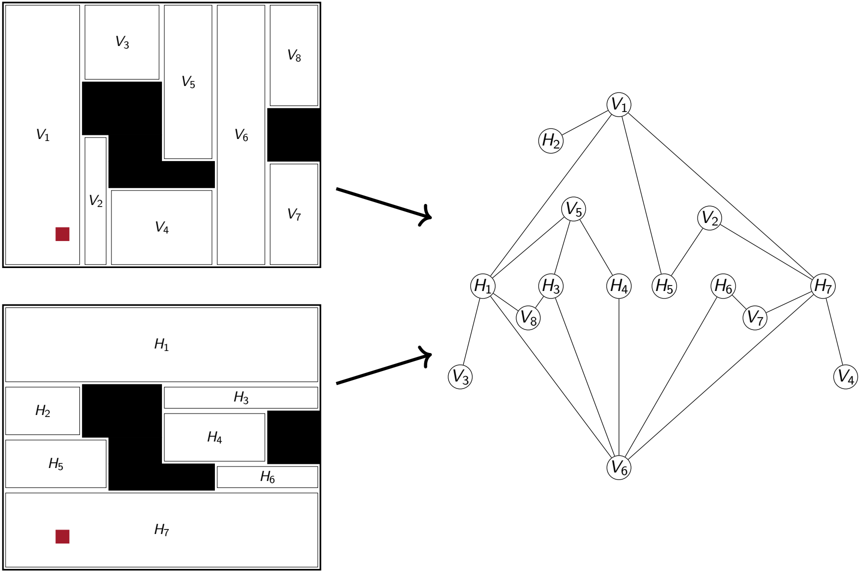

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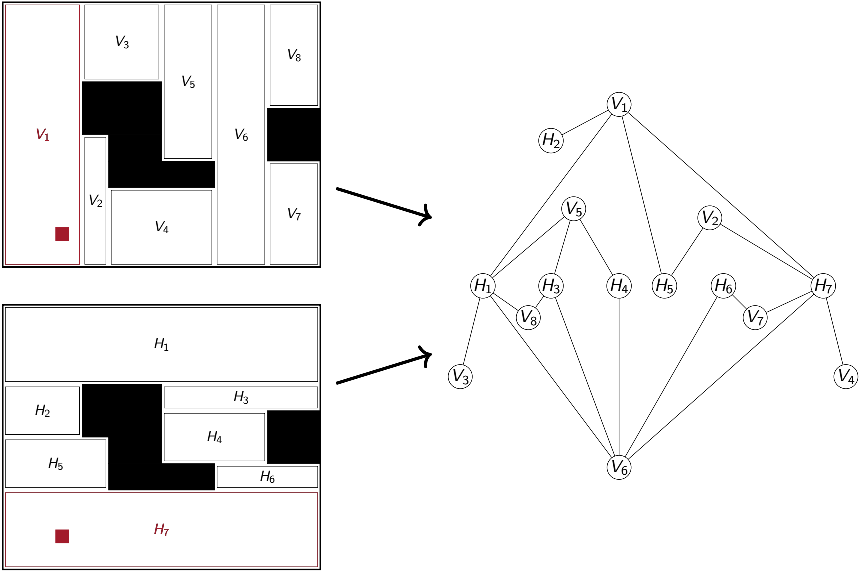

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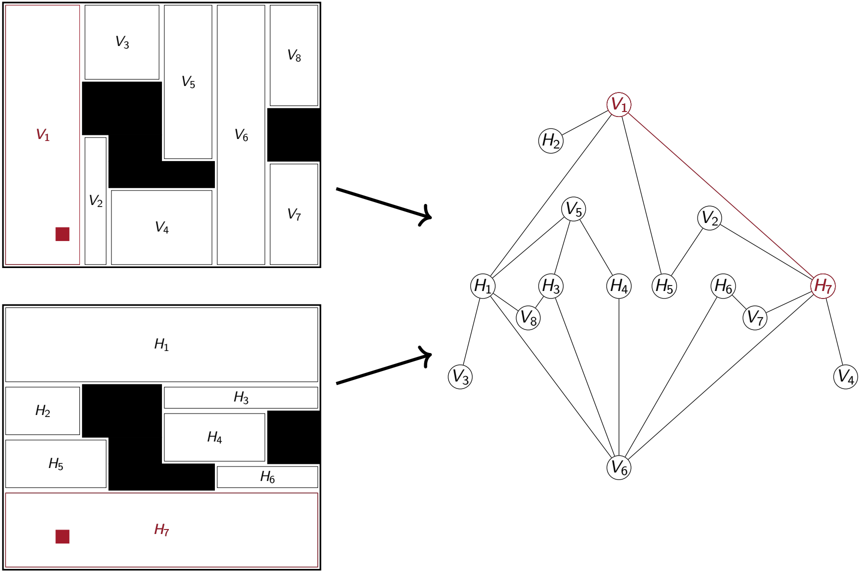

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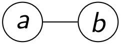

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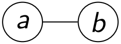

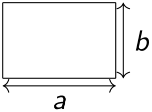

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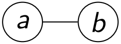

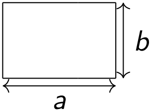

$\mathcal{G} = min(a, b) \bmod 2$

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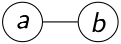

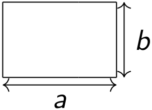

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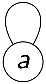

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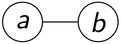

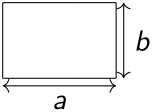

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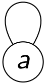

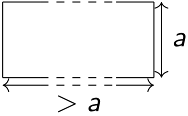

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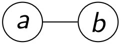

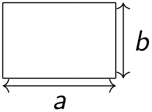

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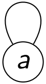

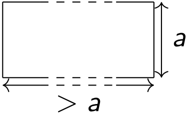

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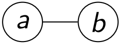

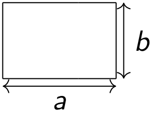

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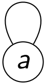

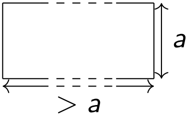

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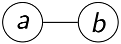

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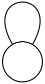


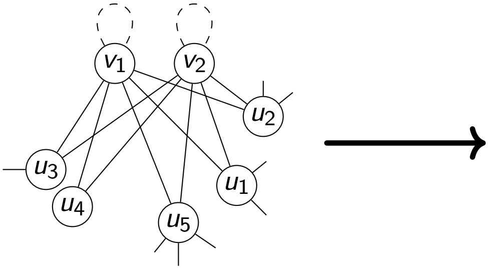
$min(a, b)$

Reduction Lemmas

Reduction Lemmas

Twin vertices lemma

If two vertices are exact false twins (including loop edges),



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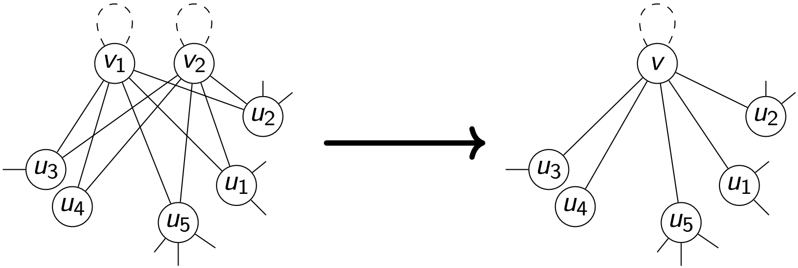

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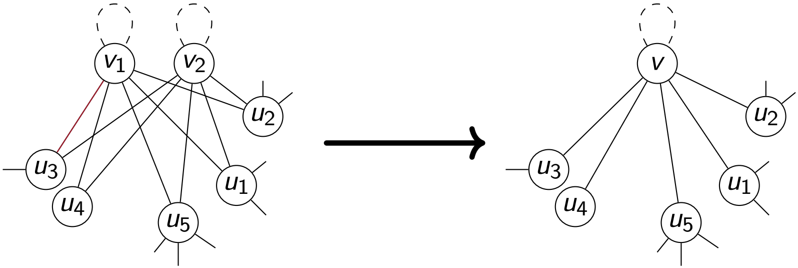

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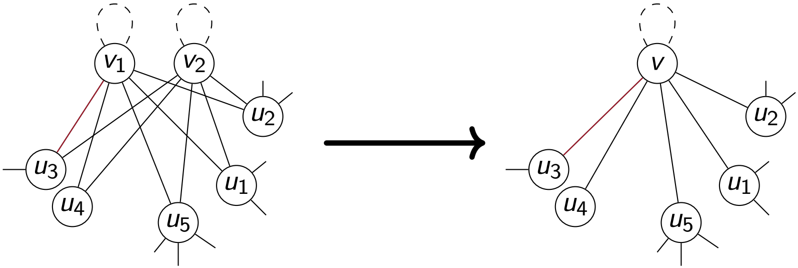

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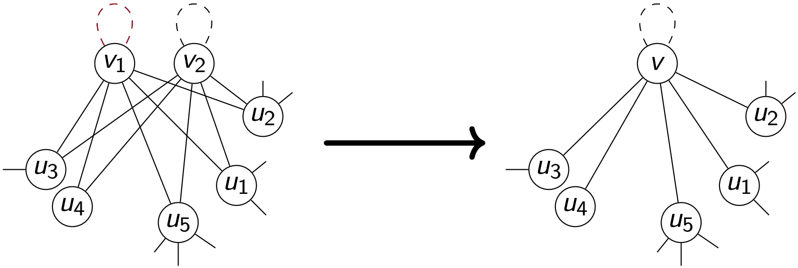

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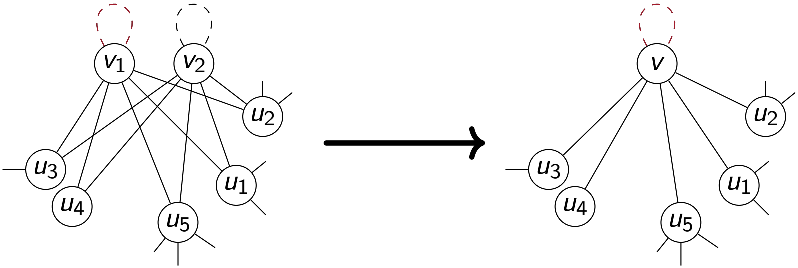

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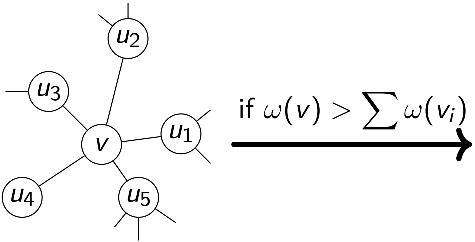

Reduction Lemmas

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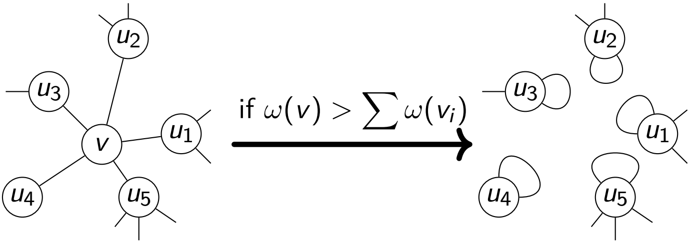
If a vertex without loop has a weight greater than the sum of its neighbour's weights, then it can be removed and loops added to its neighbours.



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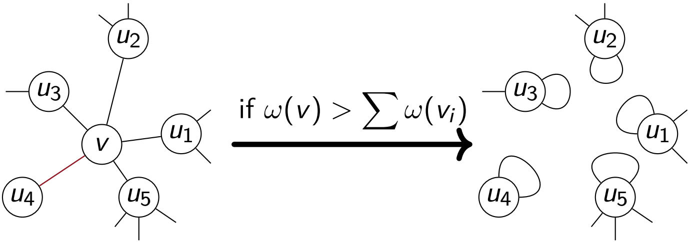

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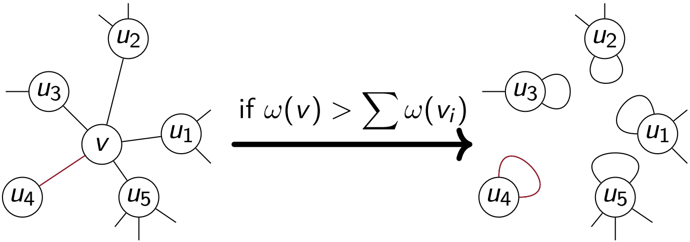

Reduction Lemmas

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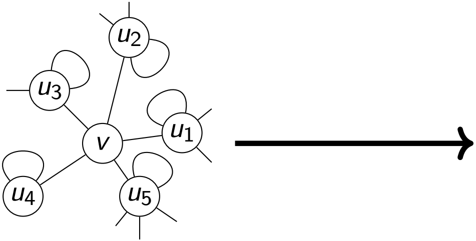
Useless vertex lemma

If all the neighbours of a vertex without loop have a loop,



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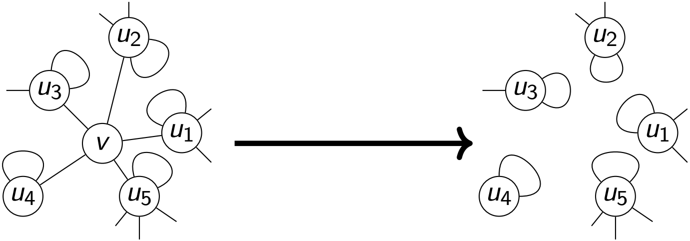

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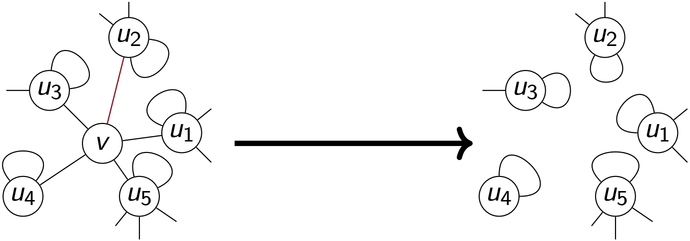

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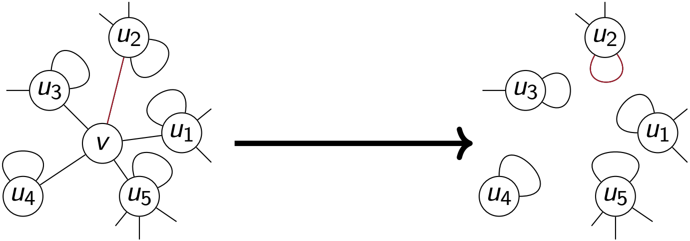
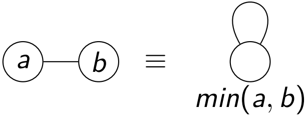


Canonical form



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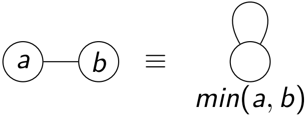

⇒ Application of the Heavy vertex lemma

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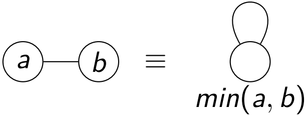

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A graph is canonical if it has no false twin, no heavy vertex and no useless vertex.

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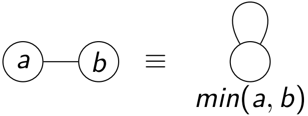

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Proposition

If G is a graph and H its canonical form after application of the reduction lemmas, then $\mathcal{G}(G) = \mathcal{G}(H)$.

What about unweighted Arc-Kayles?

Arc-Kayles (Schaefer, 1978)

- This game is played on a graph $G = (V, E)$.
- The players alternate selecting edges.
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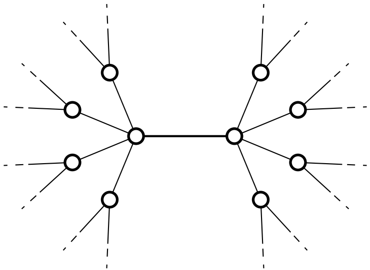

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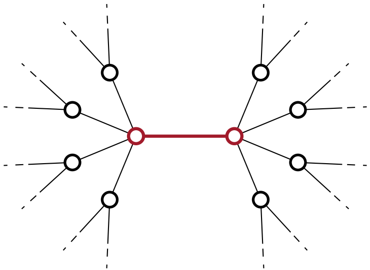

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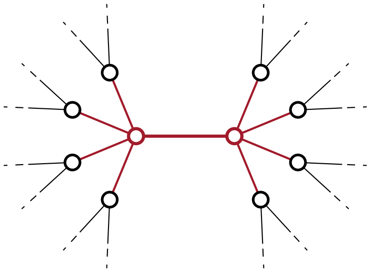

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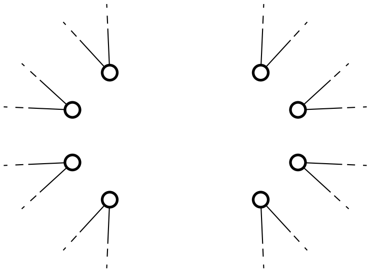

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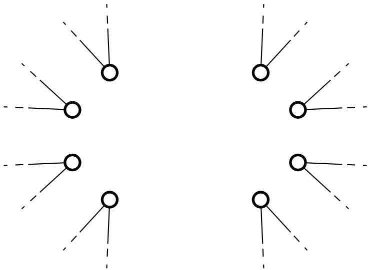

$\Rightarrow$ Arc-Kayles is WAK with $\omega(u) = 1$ for all vertex u

ARC-KAYLES: a history

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→ Links with many other games (Cram, octal games ...)

From WAK to ARC-KAYLES

WEIGHTED ARC-KAYLES

ARC-KAYLES

From WAK to Arc-Kayles

Weighted Arc-Kayles

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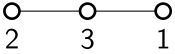

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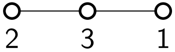
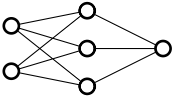


$\Longleftrightarrow$

Arc-Kayles



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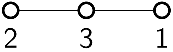

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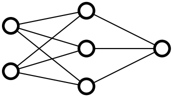

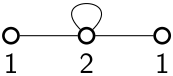

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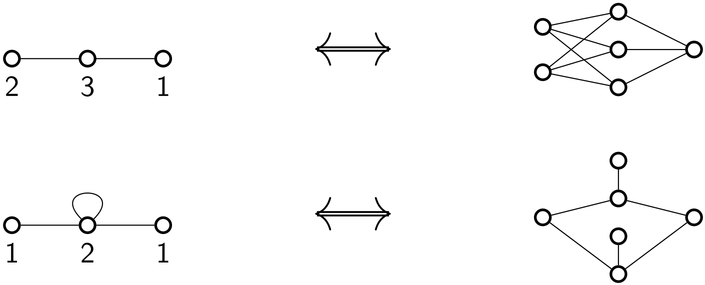
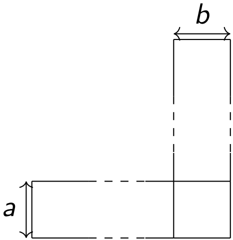


A chessboard with a hole in the corner



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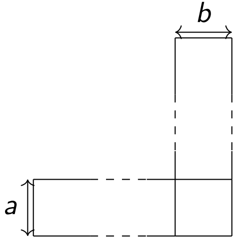

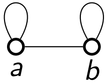

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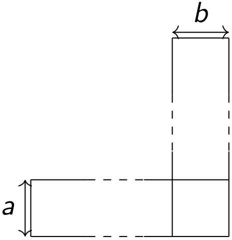

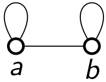

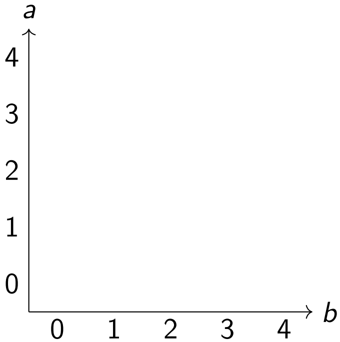

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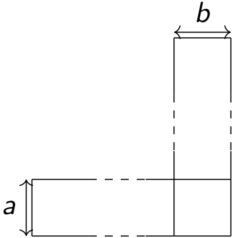

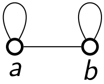

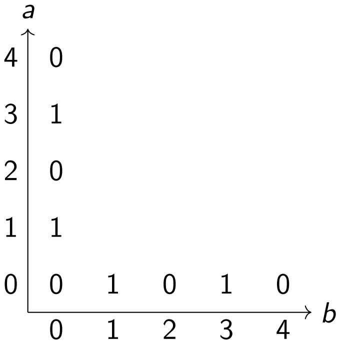

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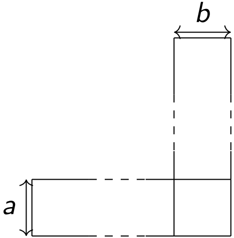

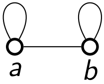

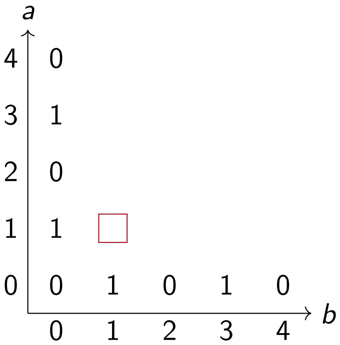

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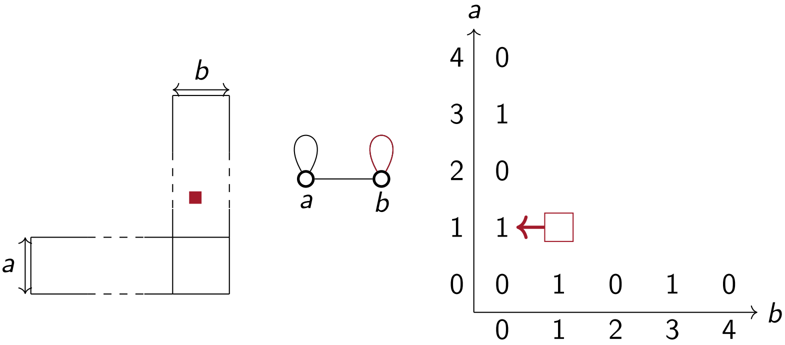

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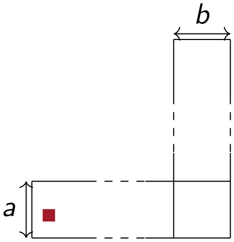

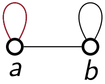

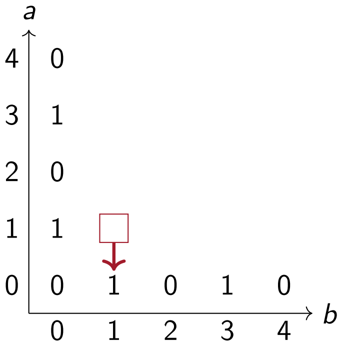

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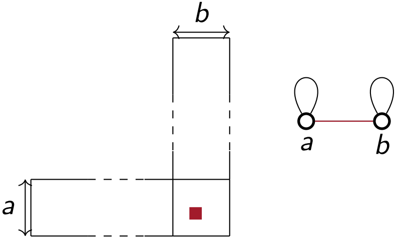

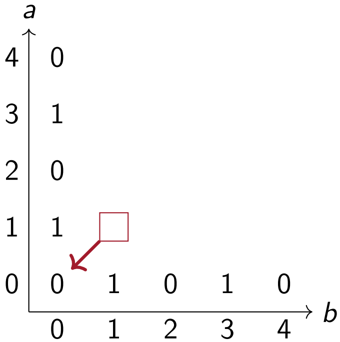

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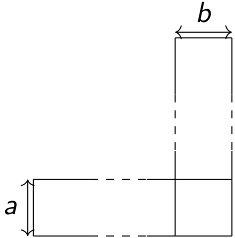

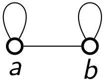

a \ b	0	1	2	3	4
4	0	3			
3	1	2			
2	0	3			
1	1	2	3	2	3
0	0	1	0	1	0

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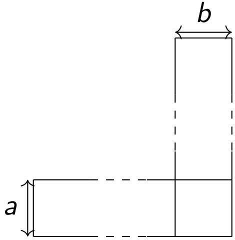

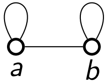

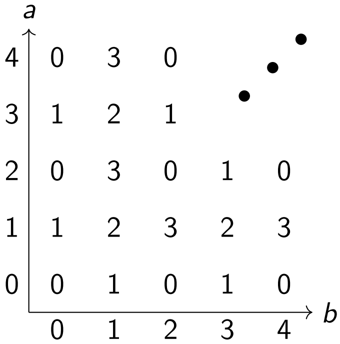

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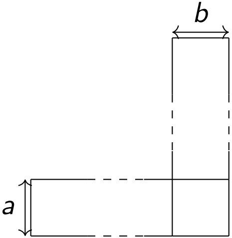
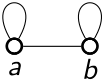
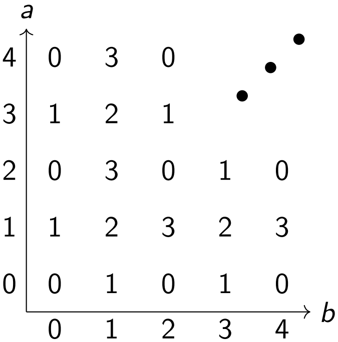




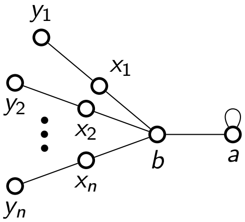


$\mathcal{P} \Leftrightarrow a$ and b even

Trees of depth at most 2

Theorem

Let $x_i > y_i$



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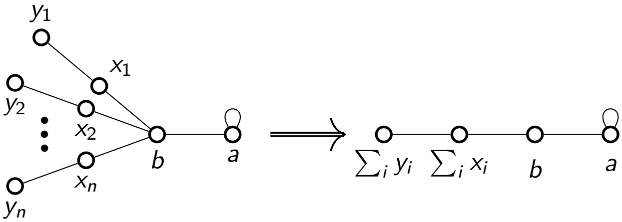

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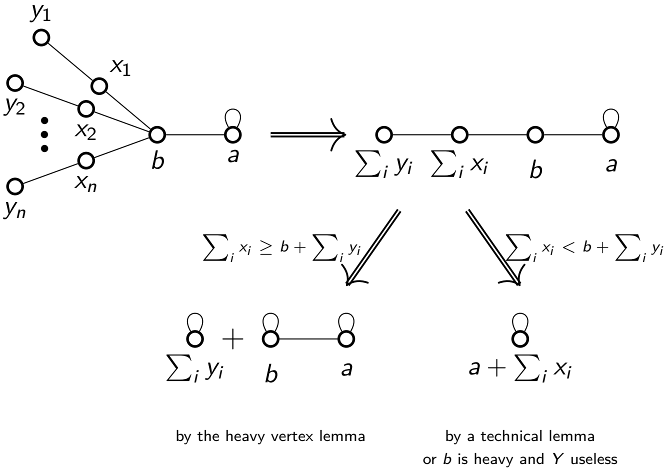

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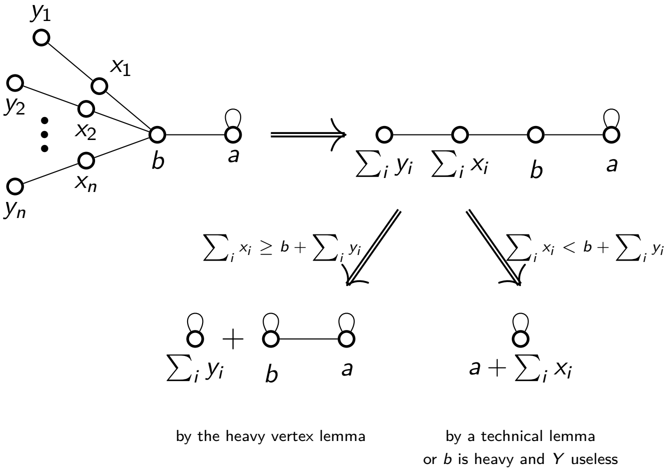
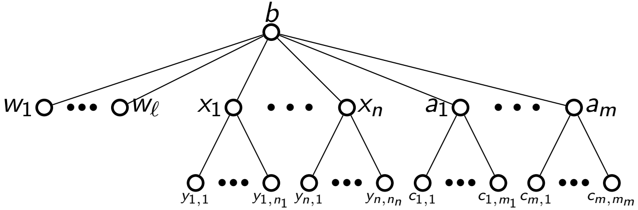


Same outcome, not equivalence!

Trees of depth at most 2

Theorem

There is a polynomial-time algorithm computing the outcome of a tree of depth at most 2.



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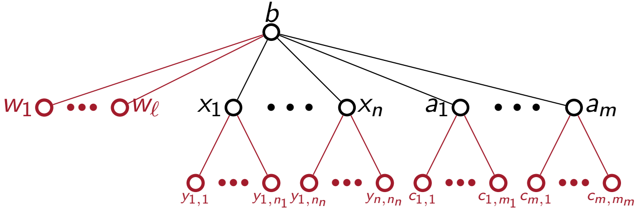

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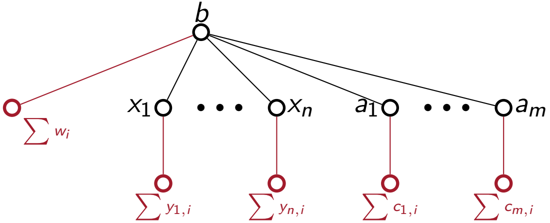

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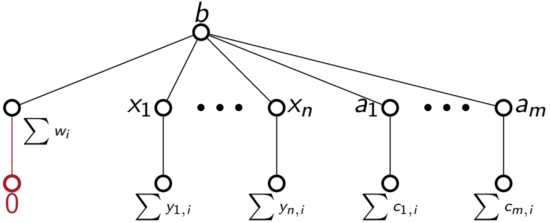

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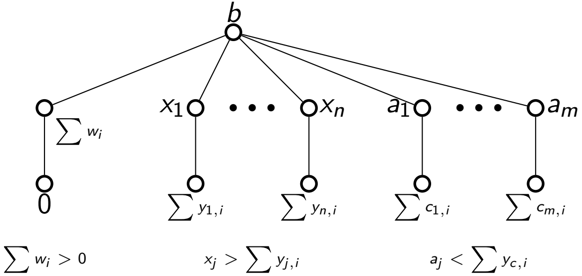

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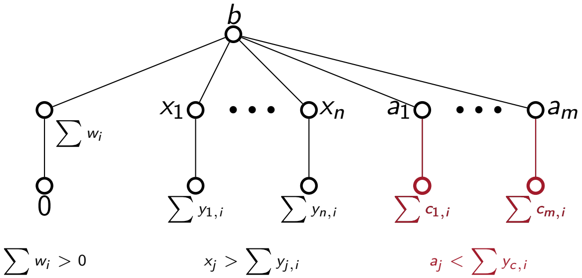

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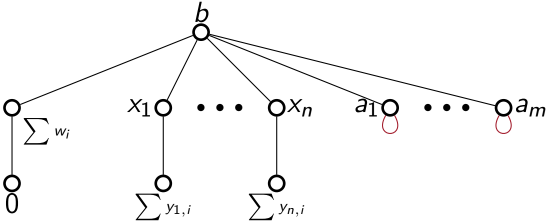

$\sum w_i > 0$ $\qquad$ $x_j > \sum y_{j,i}$

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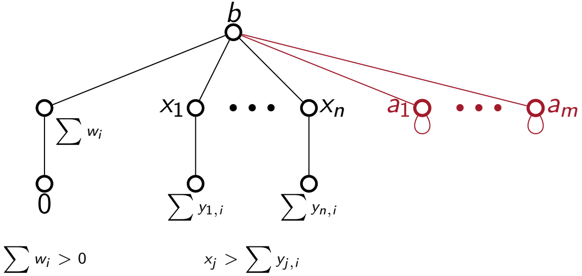

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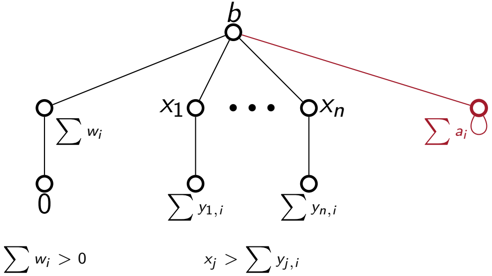

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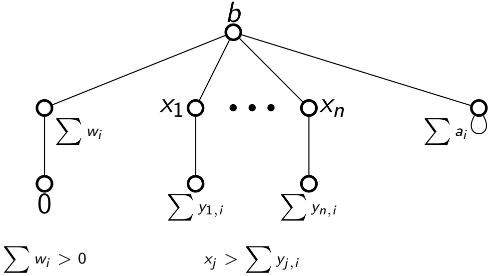
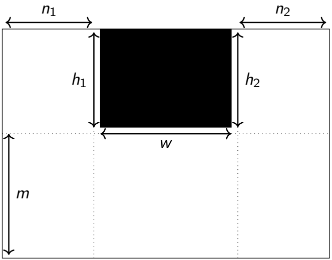


$\rightarrow$ Now we can apply the previous theorem and find the outcome

Trees of depth at most 2: implication



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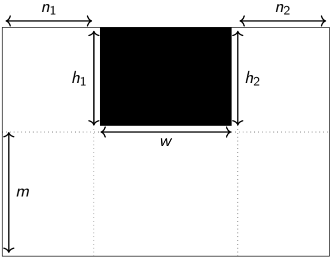
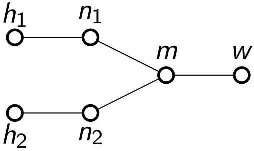


$\equiv$



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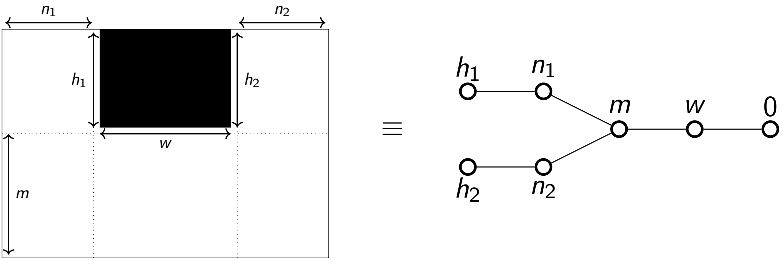
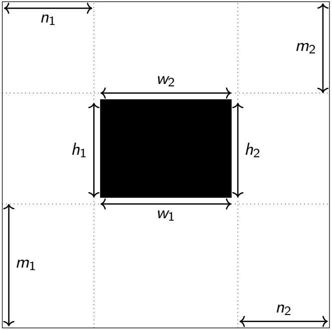


→ Tree of depth 2

Other chessboards?



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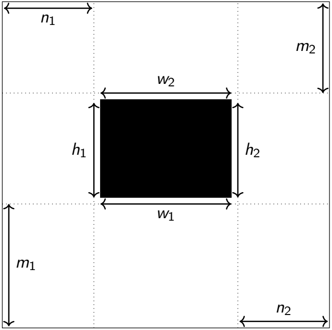

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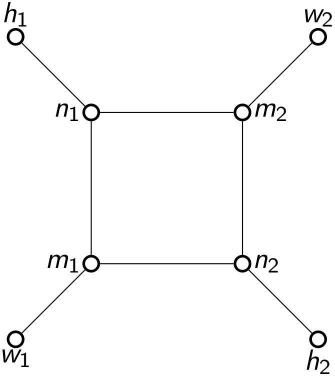

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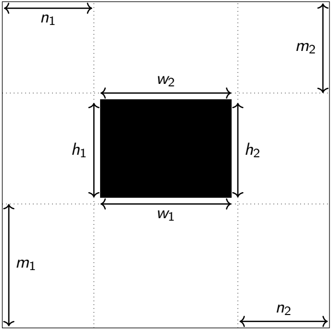

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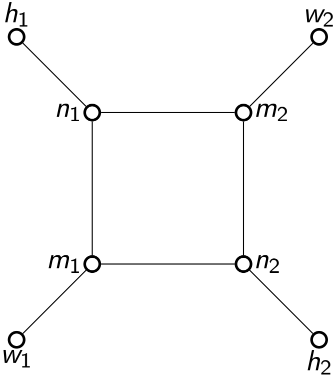

$\rightarrow$ Hard...

Are the Grundy values bounded?

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Theorem

The Grundy values for WAK are unbounded.

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The Grundy values for WAK are unbounded.

Proof (by induction)

Construct a sequence $G_1, G_2, \ldots$ such that:

- $\mathcal{G}(G_i) \neq \mathcal{G}(G_j)$ for $j < i$
- A winning move is by removing a certain vertex u_i
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$G_1 =$ 

Unboundedness of Grundy values: building G_{n+1}

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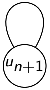

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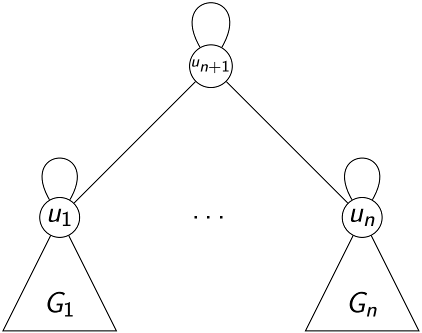

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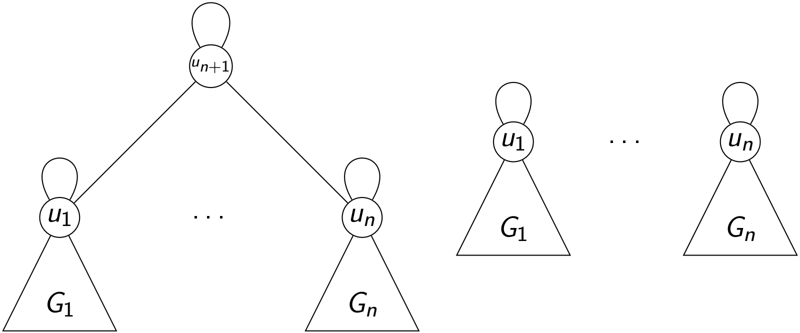

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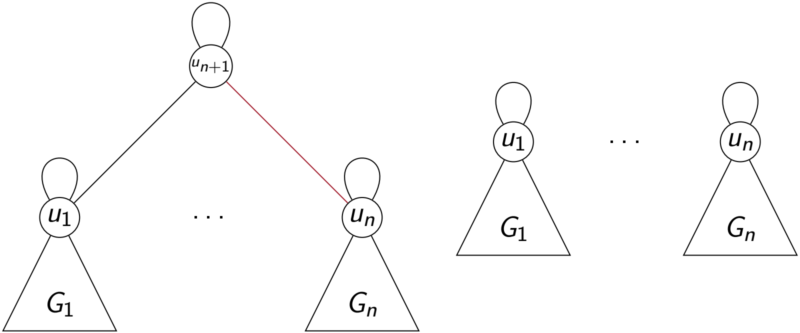

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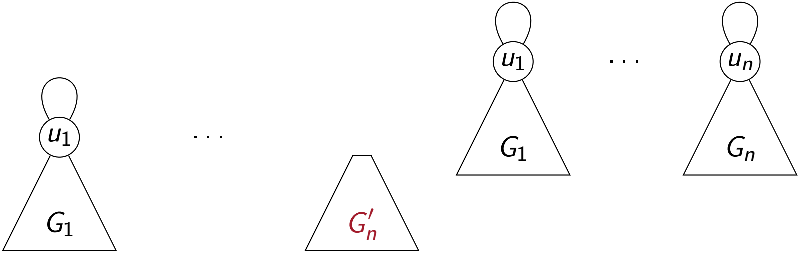

$\forall i, \mathcal{G}(G'_i) = 0$ by induction hypothesis,

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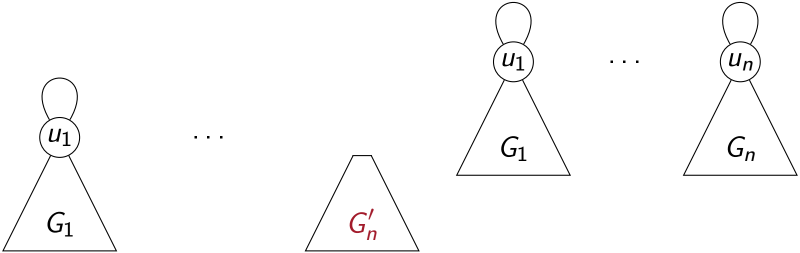

$\forall i, \mathcal{G}(G'_i) = 0$ by induction hypothesis, this graph has $\mathcal{G} = \mathcal{G}(G_i)$.

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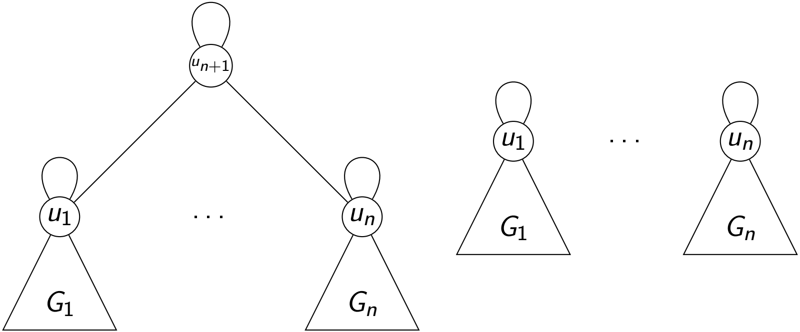

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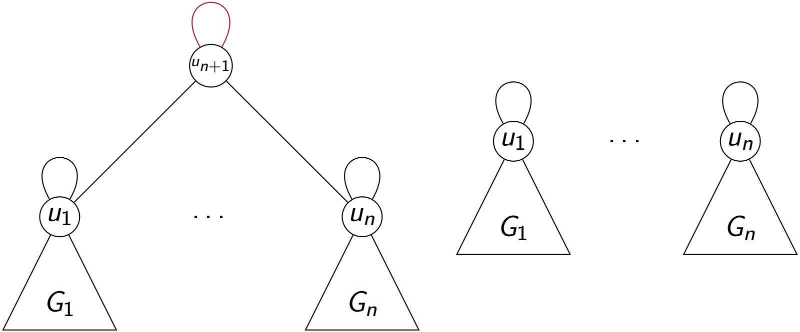

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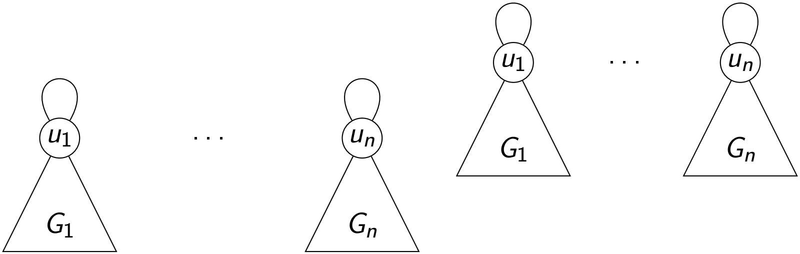

This graph has $\mathcal{G} = 0$.

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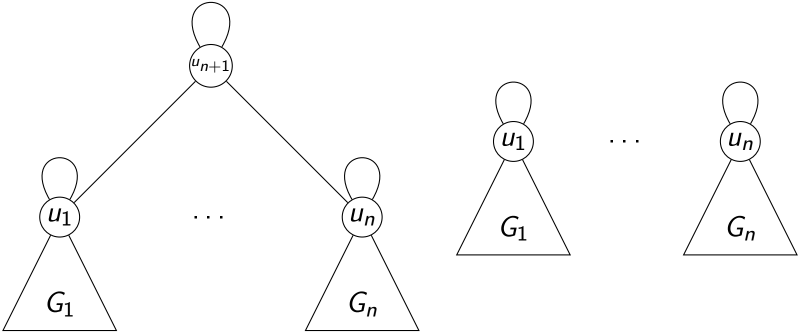

$\Rightarrow$ Infinite sequence of graphs with distinct Grundy values.

Unboundedness of Grundy values

Theorem

The Grundy values for WAK are unbounded.

Unboundedness of Grundy values

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The Grundy values for WAK are unbounded.

Corollary

The Grundy values for Arc-Kayles are unbounded.
The Grundy values for Node-Kayles are unbounded.

Conclusion

Results

- Introduction of WAK, links with the rooks game
- Outcome of WAK on trees of depth at most 2
- Unboundedness of Grundy values

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